

# Comments on complete actions for open superstring field theory

Hiroaki Matsunaga\*

\*Yukawa Institute of Theoretical Physics, Kyoto University,  
Kyoto 606-8502, Japan

## Abstract

We clarify a Wess-Zumino-Witten-like structure including Ramond fields and propose one systematic way to construct gauge invariant actions: Wess-Zumino-Witten-like complete action  $S_{\text{WZW}}$ . We show that Kunitomo-Okawa's action proposed in arXiv:1508.00366 can obtain a topological parameter dependence of Ramond fields and belongs to our WZW-like framework. In this framework, once a WZW-like functional  $\mathcal{A}_\eta = \mathcal{A}_\eta[\Psi]$  of a dynamical string field  $\Psi$  is constructed, we obtain one realization of  $S_{\text{WZW}}$  parametrized by  $\Psi$ . On the basis of this way, we construct an action  $\tilde{S}$  whose on-shell condition is equivalent to the Ramond equations of motion proposed in arXiv:1506.05774. Using these results, we provide the equivalence of two these theories: arXiv:1508.00366 and arXiv:1506.05774.

---

\*hiroaki.matsunaga@yukawa.kyoto-u.ac.jp

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Complete action and topological $t$ -dependence . . . . .	2
<b>2</b>	<b>Wess-Zumino-Witten-like complete action</b>	<b>7</b>
2.1	WZW-like structure and $XY$ -projection . . . . .	7
2.2	WZW-like complete action . . . . .	9
2.3	Unified notation . . . . .	12
<b>3</b>	<b>Another parametrization</b>	<b>13</b>
3.1	Another parametrization of the WZW-like complete action . . . . .	14
3.2	WZW-like relation from $A_\infty$ and $\eta$ -exactness . . . . .	16
3.3	Equivalence of EKS and KO theories . . . . .	20
<b>4</b>	<b>Conclusion</b>	<b>21</b>
<b>A</b>	<b>Basic facts and some identities</b>	<b>22</b>

## 1 Introduction

Recently, a field theoretical formulation of superstrings has been moved toward its new phase: An action and equations of motion including the Neveu-Schwarz and Ramond sectors were constructed [1, 2]. With recent developments [3–10], we have gradually obtained new and certain understandings of superstring field theories. In the work of [1], a gauge invariant action including the NS and R sectors was constructed without introducing auxiliary Ramond fields or self-dual constraints. They started with the Wess-Zumino-Witten-like action<sup>1</sup> of the NS Berkovits theory [12] and coupled it to the R string field  $\Psi^R$  in the restricted small Hilbert space:  $XY\Psi^R = \Psi^R$ . The dynamical string field is an amalgam of the NS large field  $\Phi^{NS}$  and the R restricted small<sup>2</sup> field  $\Psi^R$ . While the complete action [1] is given by one extension of WZW-like formulation [12–17], the other one, the Ramond equations of motion [2], is a natural extension of  $A_\infty$  formulation for the NS sector [23, 24]. The  $A_\infty$  formulation provides a systematic regularization procedure of superstring field theory [25, 26] in the early days. This procedure was extended to the case including Ramond fields and the Ramond equations of motion was constructed by introducing the concept of Ramond number projections in [2].

In this paper, we focus on these two important works [1] and [2], and discuss some interesting properties based on Wess-Zumino-Witten-like point of view. Particularly, we investigate the following three topics and obtain some exact results.

---

<sup>1</sup>Note that the starting NS action is that of  $\mathbb{Z}_2$ -reversed theory and has gauge invariance with  $(\delta e^\Phi)e^{-\Phi} = \eta\Omega - [(\eta e^\Phi)e^{-\Phi}, \Omega] + Q\Lambda$ , which is constructed from not a equation of bosonic pure gauge [15] but a equation of  $\eta$ -constraint. See also [11] for these  $\mathbb{Z}_2$ -duals for open superstrings with stubs and closed superstrings.

<sup>2</sup>See [18–22] for other fascinating approaches using Ramond fields in the large Hilbert space.

1. We show that one can add the  $t$ -dependence of Ramond string fields into the complete action proposed in [1] and make the  $t$ -dependence of the action “topological”, which leads us a natural idea of Wess-Zumino-Witten-like structure including Ramond fields.
2. We clarify a Wess-Zumino-Witten-like structure including Ramond fields and propose a Wess-Zumino-Witten-like complete action. Then, it is proved that one can carry out all computation of our action using the properties of pure-gauge-like and associated fields only. The action proposed in [1] gives one realization of our WZW-like complete action.
3. On the basis of this WZW-like framework, we construct an action whose equations of motion gives the Ramond equations of motion proposed in [2]. As well as the action proposed in [1], this action also gives another realization of our WZW-like complete action: different parameterization of the same WZW-like structure and action.

These facts provide the equivalence of two (WZW-like) theories [1] and [2] on the basis of the same discussion demonstrated in [3]. Then, we can also read the relation giving a field redefinition of NS and R string fields with a partial gauge fixing or a trivial uplift by the same way used in [3, 4] or [5] for the NS sector of open superstrings without stubs.

This paper is organized as follows. First, we introduce a  $t$ -dependence of Ramond string fields and transform the complete action proposed in [1] into the form which has topological  $t$ -dependence in section 1.1. Then, we clarify a Wess-Zumino-Witten-like structure including Ramond fields. In section 2, we propose a Wess-Zumino-Witten-like complete action. We show that our WZW-like complete action has so-called topological parameter dependence in section 2.1 and is gauge invariant in 2.2. In particular, these properties all can be proved by computations based only on the properties of pure-gauge-like fields and associated fields, which is a key point of our construction. In other words, to obtain the variation of the action, equations of motion, and gauge invariance, one does NOT need explicit form or detailed properties of  $F$  giving  $F\eta F^{-1} = D_\eta^{\text{NS}}$  and  $F\Xi$  satisfying  $D_\eta^{\text{NS}}F\Xi + F\Xi D_\eta^{\text{NS}} = 1$ , which would heavily depend on the set up of theory. In section 3, we construct an action reproducing the same equations of motion as that proposed in [2]. For this purpose, it is shown that a Wess-Zumino-Witten-like structure naturally arises from  $A_\infty$  relations and  $\eta$ -exactness of the small Hilbert space in section 3.2. As well as the action proposed in [1], this action also gives another realization of our WZW-like complete action. Utilizing these facts, we discuss the equivalence of two theories [1] and [2] in section 3.3. We end with some conclusions. Some proofs are in appendix A.

## 1.1 Complete action and topological $t$ -dependence

In this section, we use the same notation as [1]. First, we show that one can add a parameter dependence of R string fields into Kunitomo-Okawa’s action, and that the resultant action has topological parameter dependence. Next, from these computations, we identify a pure-gauge-like field  $A_\eta^{\text{R}}$  and an associated field  $A_d^{\text{R}}$  of the Ramond sector. We end this section by introducing a Wess-Zumino-Witten-like form of Kunitomo-Okawa’s action.

### State space and $XY$ -restriction

First, we introduce the large and small Hilbert spaces. The large Hilbert space  $\mathcal{H}$  is the state space whose superconformal ghost sector is spanned by  $\xi(z)$ ,  $\eta(z)$ , and  $\phi(z)$ . We write  $\eta$  for the zero mode  $\eta_0$  and  $\mathcal{H}_S$  for the kernel of  $\eta \equiv \eta_0$ . We call this subspace  $\mathcal{H}_S \subsetneq \mathcal{H}$  the small Hilbert space, whose superconformal ghost sector is spanned by  $\beta(z)$  and  $\gamma(z)$ . Let  $P_\eta$  is a projector onto the  $\eta$ -exact states: we can write  $\mathcal{H}_S = P_\eta \mathcal{H}$  because  $\eta$ -complex is exact in  $\mathcal{H}$ . Following the commutation relation  $\eta\xi = 1 - \xi\eta$  for  $\xi = \xi_0$  or  $\Xi$  of [1], we define a projector  $P_\xi \equiv 1 - P_\eta$  onto the not  $\eta$ -exact states. Note also that for any state  $\Phi \in \mathcal{H}$ , these projectors act as

$$P_\eta + P_\xi = 1, \quad P_\eta^2 = P_\eta, \quad P_\xi^2 = P_\xi, \quad P_\eta P_\xi = P_\xi P_\eta = 0,$$

by definition, and that  $P_\eta$  acts as the identity operator 1 on  $\Phi \in \mathcal{H}_S$  because of  $\mathcal{H}_S \subset P_\eta \mathcal{H}_S$ .

Next, we consider the restriction of the state space. Let  $X$  be a picture-changing operator which is a Grassmann even and picture number 1 operator defined by  $X = \delta(\beta_0)G_0 - b_0\delta'(\beta_0)$ , and let  $Y$  be an inverse picture-changing operator which is a Grassmann even and picture number  $-1$  operator defined by  $Y = c_0\delta'(\gamma_0)$ . These operator satisfy

$$XYX = X, \quad YXY = Y, \quad QX = XQ, \quad \eta X = X\eta.$$

The restricted space is the state space spanned by the states  $\Psi \in \mathcal{H}$  satisfying  $XY\Psi = \Psi$ , on which the operator  $XY$  becomes a projector  $(XY)^2 = XY$ . The restricted small space  $\mathcal{H}_R$  is the space spanned states  $\Psi$  satisfying

$$XY\Psi = \Psi, \quad \eta\Psi = 0.$$

We use this restricted small Hilbert space  $\mathcal{H}_R$  as the state space of the Ramond string field. See also [27–30]. One can quickly check that when  $\Psi$  is in  $\mathcal{H}_R$ ,  $Q\Psi$  is also in  $\mathcal{H}_R$ . See (2.25) of [1].

### Action

Let  $\Phi^{\text{NS}}$  be a Neveu-Schwarz open string field of the Berkovits theory, which is a Grassmann even and ghost-and-picture number  $(0|0)$  state in the large Hilbert space  $\mathcal{H}$ , and let  $\Psi^{\text{R}}$  be a Ramond open string field of [1], which is a Grassmann odd and ghost-and-picture number  $(1|-\frac{1}{2})$  state in the restricted small Hilbert space  $\mathcal{H}_R$ . The kinetic term is given by

$$S_0 = -\frac{1}{2}\langle\Phi^{\text{NS}}, Q\eta\Phi^{\text{NS}}\rangle - \frac{1}{2}\langle\xi Y\Psi^{\text{R}}, Q\Psi^{\text{R}}\rangle,$$

where  $Q$  is the BRST operator of open superstrings,  $\langle A, B \rangle$  is the BPZ inner product of  $A, B \in \mathcal{H}$  in the large Hilbert space  $\mathcal{H}$ . As explained in [1], we can use both  $\xi = \xi_0$  and  $\xi = \Xi$  for the above  $\xi$  in the BPZ inner product. Utilizing NS WZW-like functionals  $A_\eta^{\text{NS}} = (\eta e^{\Phi^{\text{NS}}})e^{-\Phi^{\text{NS}}}$ ,  $A_t^{\text{NS}} = (\partial_t e^{\Phi^{\text{NS}}})e^{-\Phi^{\text{NS}}}$  and the invertible linear map  $F$ , the full action is given by

$$S = -\frac{1}{2}\langle\xi Y\Psi^{\text{R}}, Q\Psi^{\text{R}}\rangle - \int_0^1 dt \langle A_t^{\text{NS}}(t), Q A_\eta^{\text{NS}}(t) + m_2(F(t)\Psi^{\text{R}}, F(t)\Psi^{\text{R}})\rangle,$$

where  $m_2$  is the Witten's star product  $m_2(A, B) \equiv A * B$  [31] and as well as  $A_\eta^{\text{NS}}(t)$  or  $A_t^{\text{NS}}(t)$ ,  $F(t)$  also satisfies  $F(t=0) = 0$  and  $F(t=1) = F$ . Note that  $F$  has no ghost-and-picture number and satisfies  $F\eta F^{-1} = D_\eta^{\text{NS}}$  and  $D_\eta^{\text{NS}}F\Xi + F\Xi D_\eta^{\text{NS}} = 1$ . In this paper, we do not need the explicit form of  $F$ . See [1] or appendix A for the explicit form of  $F$ .

In this Kunitomo-Okawa's action, only the NS field  $\Phi^{\text{NS}}(t)$  has a parameter dependence and the R field  $\Phi^{\text{R}}$  has not:  $\partial_t \Phi^{\text{NS}}(t) \neq 0$  and  $\partial_t \Psi^{\text{R}} = 0$ , where a  $t$ -parametrized NS field  $\Phi^{\text{NS}}(t)$  is a path on the state space satisfying  $\Phi^{\text{NS}}(t=0) = 0$  and  $\Phi^{\text{NS}}(t=1) = \Phi^{\text{NS}}$ . We show that a complete action of open superstring field theory proposed in [1] can be written as

$$S = - \int_0^1 dt \left( \langle \xi Y \partial_t \Psi^{\text{R}}(t), QF(t) \Psi^{\text{R}}(t) \rangle + \langle A_t^{\text{NS}}(t), QA_\eta^{\text{NS}}(t) + m_2(F(t) \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t)) \rangle \right) \quad (1.1)$$

using a  $t$ -parametrized R field  $\Psi^{\text{R}}(t)$  satisfying  $\Psi^{\text{R}}(t=0) = 0$  and  $\Psi^{\text{R}}(t=1) = \Psi^{\text{R}}$ . Then, we also show that  $t$ -dependence of (1.1) is topological

$$\delta S = - \langle \xi Y \delta \Psi^{\text{R}}, QF \Psi^{\text{R}} \rangle - \langle A_\delta^{\text{NS}}, QA_\eta^{\text{NS}} + m_2(F \Psi^{\text{R}}, F \Psi^{\text{R}}) \rangle. \quad (1.2)$$

### Topological $t$ -dependence

Let  $P_\eta$  be a projector onto the space of  $\eta$ -exact states and let  $P_\xi$  be a projector defined by  $P_\xi \equiv 1 - P_\eta$ . For example, one can use  $P_\eta = \eta \xi$  and  $P_\xi = \xi \eta$ . Note that these projectors  $P_\eta$  and  $P_\xi$  satisfy  $P_\eta + P_\xi = 1$  on the large Hilbert space  $\mathcal{H}$ . One can check that

$$\begin{aligned} \langle \xi Y \partial_t \Psi^{\text{R}}(t), QF(t) \Psi^{\text{R}}(t) \rangle &= \langle \xi Y \partial_t \Psi^{\text{R}}(t), Q(P_\eta + P_\xi)F(t) \Psi^{\text{R}}(t) \rangle \\ &= \langle \xi Y \partial_t \Psi^{\text{R}}(t), Q \Psi^{\text{R}}(t) \rangle + \langle \xi Y \partial_t \Psi^{\text{R}}(t), \eta X F(t) \Psi^{\text{R}}(t) \rangle \\ &= \frac{\partial}{\partial t} \left( \frac{1}{2} \langle \xi Y \Psi^{\text{R}}(t), Q \Psi^{\text{R}}(t) \rangle \right) - \langle \partial_t \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t) \rangle \end{aligned} \quad (1.3)$$

using  $P_\eta(F \Psi^{\text{R}}) = \Psi^{\text{R}}$  and  $P_\eta Q P_\xi = \eta X$ . See appendix A for the BPZ properties.

Let  $\llbracket A, B \rrbracket$  be a graded commutator of the star product  $m_2$ :

$$\llbracket A, B \rrbracket \equiv m_2(A, B) - (-)^{AB} m_2(B, A).$$

The upper index of  $(-)^A$  denotes the grading of the state  $A$ , namely, its ghost number. Then, we can write  $\llbracket F \Psi^{\text{R}}, F \Psi^{\text{R}} \rrbracket = 2m_2(F \Psi^{\text{R}}, F \Psi^{\text{R}})$  and  $\llbracket D_\eta^{\text{NS}}, F \Xi \rrbracket = 1$ . Some computations are in appendix A. Similarly, as (1.3), using the  $D_\eta^{\text{NS}}$ -exactness  $F \Psi^{\text{R}} = D_\eta^{\text{NS}} F \Xi \Psi^{\text{R}}$  and

$$\partial_t(F(t) \Psi^{\text{R}}(t)) = F(t) \partial_t \Psi^{\text{R}}(t) + F(t) \Xi D_\eta^{\text{NS}}(t) \llbracket A_t^{\text{NS}}(t), F(t) \Psi^{\text{R}}(t) \rrbracket,$$

one can also check that

$$\begin{aligned} \langle A_t^{\text{NS}}, m_2(F(t) \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t)) \rangle &= -\frac{1}{2} \langle F(t) \Psi^{\text{R}}(t), \llbracket A_t^{\text{NS}}(t), F(t) \Psi^{\text{R}}(t) \rrbracket \rangle \\ &= -\frac{1}{2} \langle \Psi^{\text{R}}(t), F(t) \Xi D_\eta^{\text{NS}}(t) \llbracket A_t^{\text{NS}}(t), F(t) \Psi^{\text{R}}(t) \rrbracket \rangle \\ &= -\frac{1}{2} \langle \Psi^{\text{R}}(t), \partial_t(F(t) \Psi^{\text{R}}(t)) - F(t) \partial_t \Psi^{\text{R}}(t) \rangle. \end{aligned}$$

Furthermore, since  $\eta \Psi^{\text{R}} = 0$  and thus  $\langle \partial_t \Psi^{\text{R}}, F \Psi^{\text{R}} \rangle + \langle F \partial_t \Psi^{\text{R}}, \Psi^{\text{R}} \rangle = \langle \partial_t \Psi^{\text{R}}, \Psi^{\text{R}} \rangle = 0$ , we find

$$\begin{aligned} \langle A_t^{\text{NS}}(t), m_2(F(t) \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t)) \rangle &= -\frac{1}{2} \langle \Psi^{\text{R}}(t), \partial_t(F(t) \Psi^{\text{R}}(t)) \rangle + \frac{1}{2} \langle \partial_t \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t) \rangle \\ &= \frac{\partial}{\partial t} \left( -\frac{1}{2} \langle \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t) \rangle \right) + \langle \partial_t \Psi^{\text{R}}(t), F(t) \Psi^{\text{R}}(t) \rangle. \end{aligned} \quad (1.4)$$

Therefore using  $\langle \Psi^R, F\Psi^R \rangle = -\langle \xi Y \Psi^R, \eta X F \Psi^R \rangle$ , we obtain

$$(1.3) + (1.4) = \frac{1}{2} \left( \langle \xi Y \Psi^R, Q\Psi^R + \eta X F \Psi^R \rangle \right) = \frac{1}{2} \langle \xi Y \Psi^R, QF\Psi^R \rangle.$$

As a result, our  $t$ -integrated action (1.1) becomes

$$\begin{aligned} S &= -\frac{1}{2} \langle \xi Y \Psi^R, QF\Psi^R \rangle - \int_0^1 dt \langle A_t^{\text{NS}}(t), Q A_\eta^{\text{NS}}(t) \rangle, \\ &= -\frac{1}{2} \langle \xi Y \Psi^R, Q\Psi^R \rangle - \int_0^1 dt \langle A_t^{\text{NS}}(t), Q A_\eta^{\text{NS}}(t) + m_2(F(t)\Psi^R, F(t)\Psi^R) \rangle. \end{aligned} \quad (1.5)$$

The second line is the original form used in [1] but we do not need the second line expression to show that the variation of the action (1.1) is given by (1.2). Translation from the second line to the first line is in appendix A. Since the variation of the first term of the first line in (1.5) is

$$\delta \left( \frac{1}{2} \langle \xi Y \Psi^R, QF\Psi^R \rangle \right) = \langle \xi Y \delta \Psi^R, QF\Psi^R \rangle + \langle A_\delta^{\text{NS}}, m_2(F\Psi^R, F\Psi^R) \rangle,$$

we obtain (1.2) and the action  $S$  has topological  $t$ -dependence.

### Gauge invariance

Let  $\Omega^{\text{NS}}$  and  $\Lambda^{\text{NS}}$  be ghost-and-picture number  $(-1|-1)$  and  $(-1|0)$  NS states of the large Hilbert space  $\mathcal{H}$  respectively, and let  $\lambda^R$  be a ghost-and-picture number  $(0|-\frac{1}{2})$  R state of the restricted small Hilbert space  $\mathcal{H}_R$ :  $\eta\lambda^R = 0$  and  $XY\lambda^R = \lambda^R$ . These states naturally appear in gauge transformations of the action. The action  $S$  has gauge invariances:  $\delta_{\Omega^{\text{NS}}} S = 0$  with  $\Omega^{\text{NS}}$ -gauge transformations

$$A_{\delta_{\Omega^{\text{NS}}}}^{\text{NS}} = \eta\Omega^{\text{NS}} - \llbracket A_\eta^{\text{NS}} \Omega^{\text{NS}} \rrbracket, \quad \delta_{\Omega^{\text{NS}}} \Psi^R = 0, \quad (1.6)$$

$\delta_{\Lambda^{\text{NS}}} S = 0$  with  $\Lambda^{\text{NS}}$ -gauge transformations

$$A_{\delta_{\Lambda^{\text{NS}}}}^{\text{NS}} = Q\Lambda^{\text{NS}} + \llbracket F\Psi^R, F\Xi \llbracket F\Psi^R, \Lambda^{\text{NS}} \rrbracket \rrbracket, \quad \delta_{\Lambda^{\text{NS}}} \Psi^R = X\eta F\Xi D_\eta^{\text{NS}} \llbracket F\Psi^R, \Lambda^{\text{NS}} \rrbracket,$$

and  $\delta_{\lambda^R} S = 0$  with  $\lambda^R$ -gauge transformations

$$A_{\delta_{\lambda^R}}^{\text{NS}} = -\llbracket F\Psi^R, F\Xi\lambda^R \rrbracket, \quad \delta_{\lambda^R} \Psi^R = Q\lambda^R - X\eta F\Xi\lambda^R.$$

Note also that using a new gauge parameter  $\Lambda^R$  defined by

$$\Lambda^R \equiv F\Xi \left( -\lambda^R + \llbracket F\Psi^R, \Lambda^{\text{NS}} \rrbracket \right), \quad (1.7)$$

where  $\Lambda^R$  belongs to the large Hilbert space and has ghost-and-picture number  $(-1|\frac{1}{2})$ , we obtain a simpler expression of  $\Lambda$ -gauge transformations as follows

$$A_{\delta_{\Lambda}}^{\text{NS}} = Q\Lambda^{\text{NS}} + \llbracket A_\eta^R, \Lambda^R \rrbracket, \quad (1.8)$$

$$\delta_{\Lambda} \Psi^R = -P_\eta Q \left( \eta\Lambda^R - \llbracket A_\eta^{\text{NS}}, \Lambda^R \rrbracket - \llbracket F\Psi^R, \Lambda^{\text{NS}} \rrbracket \right). \quad (1.9)$$

### Ramond pure-gauge-like field $A_\eta^R$

In the rest of this section, we identify a pure-gauge-like field  $A_\eta^R$  and associated fields  $A_d^R$  in the Ramond sector and rewrite the action (1.1) into our Wess-Zumino-Witten-like form.

We write  $A_\eta^R$  for  $F\Psi^R$ , which is one realization of a *Ramond pure-gauge-like field*:

$$A_\eta^R \equiv F\Psi^R. \quad (1.10)$$

By definition, the R pure-gauge-like field  $A_\eta^R$  satisfies  $D_\eta^{\text{NS}} A_\eta^R = 0$ , namely,

$$\eta A_\eta^R - \llbracket A_\eta^{\text{NS}}, A_\eta^R \rrbracket = 0. \quad (1.11)$$

As we will explain, the  $\eta$ -exact component  $P_\eta A_\eta^R$  appears in the action and its properties is important. Since the linear map  $F$  satisfies  $\xi F = \xi$  for  $\xi = \Xi$ , we quickly find that

$$\begin{aligned} P_\eta A_\eta^R &= \Psi^R, \\ d(P_\eta A_\eta^R) &= d\Psi^R, \end{aligned}$$

where  $P_\eta = \eta\xi$  is a projector onto the small Hilbert space  $\mathcal{H}_S$  and  $d = Q, \partial_t, \delta$  is a derivation operator commuting with  $\eta$ . Then, using  $P_\eta A_\eta^R$ , we can express the *XY-projection invariance* of Ramond string fields  $XY\Psi^R = \Psi^R$  by

$$XY(P_\eta A_\eta^R) = P_\eta A_\eta^R. \quad (1.12)$$

Note that  $P_\eta A_\eta^R \in \mathcal{H}_R$ . Similarly, we introduce a *Ramond associated field*  $A_d^R$  by

$$(-)^d A_d^R \equiv F\Xi \left( d\Psi^R - (-)^d \llbracket A_d^{\text{NS}}, F\Psi^R \rrbracket - (-)^d \eta \llbracket d, \Xi \rrbracket F\Psi^R \right). \quad (1.13)$$

Using properties of  $F$ , one can check that the R associated field  $A_d^R$  satisfies

$$(-)^d dA_\eta^R = \eta A_d^R - \llbracket A_\eta^{\text{NS}}, A_d^R \rrbracket - \llbracket A_\eta^R, A_d^{\text{NS}} \rrbracket, \quad (1.14)$$

or equivalently,  $(-)^d dA_\eta^R = D_\eta^{\text{NS}} A_d^R - \llbracket A_\eta^R, A_d^{\text{NS}} \rrbracket$ . See appendix A. Then, we obtain

$$\langle \xi Y \partial_t \Psi^R, QF\Psi^R \rangle + \langle A_t^{\text{NS}}, m_2(F\Psi^R, F\Psi^R) \rangle = \langle \xi Y \partial_t (P_\eta A_\eta^R), QA_\eta^R \rangle + \langle A_t^{\text{NS}}, m_2(A_\eta^R, A_\eta^R) \rangle.$$

Utilizing these expressions, the action becomes

$$S = - \int_0^1 dt \left( \langle \xi Y \partial_t (P_\eta A_\eta^R), QA_\eta^R \rangle + \langle A_t^{\text{NS}}, QA_\eta^{\text{NS}} + m_2(A_\eta^R, A_\eta^R) \rangle \right). \quad (1.15)$$

Note also that gauge transformation parametrized by  $\Omega = \Omega^{\text{NS}} + \Omega^R$  is given by

$$A_{\delta_\Omega}^{\text{NS}} = \eta \Omega^{\text{NS}} - \llbracket A_\eta^{\text{NS}}, \Omega^{\text{NS}} \rrbracket, \quad \delta_\Omega (P_\eta A_\eta^R) = 0.$$

and gauge transformations parametrized by  $\Lambda = \Lambda^{\text{NS}} + \Lambda^R$  is given by

$$\begin{aligned} A_{\delta_\Lambda}^{\text{NS}} &= Q\Lambda^{\text{NS}} + \llbracket A_\eta^R, \Lambda^R \rrbracket, \\ \delta_\Lambda (P_\eta A_\eta^R) &= -P_\eta Q \left( \eta \Lambda^R - \llbracket A_\eta^{\text{NS}}, \Lambda^R \rrbracket - \llbracket A_\eta^R, \Lambda^{\text{NS}} \rrbracket \right), \end{aligned}$$

where we use a R state  $\Lambda^R$ , a redefined R gauge parameter,

$$\Lambda^R \equiv F\Xi \left( -\lambda^R + \llbracket A_\eta^R, \Lambda^{\text{NS}} \rrbracket \right),$$

which belongs to the large Hilbert space  $\mathcal{H}$  and has ghost-and-picture number  $(-1|\frac{1}{2})$ .

In the work of [1], all computations of the variation of the action, equations of motion, and gauge invariance heavily depend on the explicit form or properties of the linear map  $F$ . However, as we will show in the next section, these all computations are derived from WZW-like properties of the Ramond sector: (1.11), (1.12), and (1.14).

## 2 Wess-Zumino-Witten-like complete action

We first summarize Wess-Zumino-Witten-like relations of the NS sector and the R sector separately, and show that these relations indeed provide the topological parameter dependence of the action. Second, coupling NS and R, we define a Wess-Zumino-Witten-like complete action and prove that the gauge invariance of the action is also derived from the WZW-like relations. Lastly, we introduce a notation unifying separately given results of NS and R sectors.

### 2.1 WZW-like structure and $XY$ -projection

#### Neveu-Schwarz sector

An NS pure-gauge-like field  $\mathcal{A}_\eta^{\text{NS}}$  is a ghost-and-picture number  $(1| - 1)$  state satisfying

$$\eta \mathcal{A}_\eta^{\text{NS}} - \frac{1}{2} \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_\eta^{\text{NS}} \rrbracket = 0. \quad (2.1)$$

Let  $d$  be a derivation operator satisfying  $\llbracket d, Q \rrbracket = 0$ , and let  $(d_g|d_p)$  be ghost-and-picture number of  $d$ . An NS associated field  $\mathcal{A}_d^{\text{NS}}$  is a ghost-and-picture number  $(d_g|d_p)$  state satisfying

$$\begin{aligned} (-)^d d \mathcal{A}_\eta^{\text{NS}} &= \eta \mathcal{A}_d^{\text{NS}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_d^{\text{NS}} \rrbracket \\ &\equiv D_\eta^{\text{NS}} \mathcal{A}_d^{\text{NS}}. \end{aligned} \quad (2.2)$$

By definition of (2.1) and (2.2), one can check that the relation

$$D_\eta^{\text{NS}} \left( d_1 \mathcal{A}_{d_2}^{\text{NS}} - (-)^{d_1 d_2} d_2 \mathcal{A}_{d_1}^{\text{NS}} - (-)^{d_1 d_2} \llbracket \mathcal{A}_{d_1}^{\text{NS}}, \mathcal{A}_{d_2}^{\text{NS}} \rrbracket \right) = 0 \quad (2.3)$$

holds when two derivation  $d_1$  and  $d_2$  satisfy  $\llbracket d_1, d_2 \rrbracket \equiv d_1 d_2 - (-)^{d_1 d_2} d_2 d_1 = 0$ .

Utilizing these fields, an NS action is given by

$$S^{\text{NS}} = - \int_0^1 dt \langle \mathcal{A}_t^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} \rangle. \quad (2.4)$$

It is known that the variation of the NS action is given

$$\delta S^{\text{NS}} = - \langle \mathcal{A}_\delta^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} \rangle, \quad (2.5)$$

which we call the topological parameter dependence of WZW-like action. See [3, 5, 11, 15].

#### Ramond sector

An R pure-gauge-like field  $\mathcal{A}_\eta^{\text{R}}$  is a ghost-and-picture number  $(1| - \frac{1}{2})$  state satisfying

$$\eta \mathcal{A}_\eta^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_\eta^{\text{R}} \rrbracket = 0, \quad (2.6)$$

or equivalently,  $D_\eta^{\text{NS}} \mathcal{A}_\eta^{\text{R}} = 0$ . Let  $d$  be a derivation operator satisfying  $\llbracket d, \eta \rrbracket = 0$ , and let  $(d_g|d_p)$  be ghost-and-picture number of  $d$ . An R associated field  $\mathcal{A}_d^{\text{R}}$  is a ghost-and-picture number  $(d_g|d_p + \frac{1}{2})$  state satisfying

$$(-)^d d \mathcal{A}_\eta^{\text{R}} = \eta \mathcal{A}_d^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_d^{\text{R}} \rrbracket - \llbracket \mathcal{A}_\eta^{\text{R}}, \mathcal{A}_d^{\text{NS}} \rrbracket, \quad (2.7)$$



namely,  $(-)^d d\mathcal{A}_\eta^R = D_\eta^{\text{NS}} \mathcal{A}_d^R - \llbracket \mathcal{A}_\eta^R, \mathcal{A}_d^{\text{NS}} \rrbracket$ .

As we will show, utilizing these fields and assuming  $XY$ -projection invariance of  $P_\eta \mathcal{A}_\eta^R$

$$XY(P_\eta \mathcal{A}_\eta^R) = P_\eta \mathcal{A}_\eta^R, \quad (2.8)$$

one can construct a gauge invariant action Wess-Zumino-Witten-likely, whose parameter dependence is topological. We propose that an R action is given by

$$S^R = - \int_0^1 dt \left( \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle + \langle \mathcal{A}_t^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle \right). \quad (2.9)$$

This  $S^R$  is Wess-Zumino-Witten-like. In other words,  $S^R$  has topological  $t$ -dependence

$$\delta S^R = - \langle \xi Y \delta(P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle - \langle \mathcal{A}_\delta^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle. \quad (2.10)$$

### Topological $t$ -dependence of $S^R$

First, we consider the variation of the first term of  $S^R$ . This term consists of two ingredients:

$$\int_0^1 dt \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle = \int_0^1 dt \left( \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q(P_\eta \mathcal{A}_\eta^R) \rangle - \langle \partial_t (P_\eta \mathcal{A}_\eta^R), \mathcal{A}_\eta^R \rangle \right).$$

We can quickly find that the first part has topological  $t$ -dependence

$$\begin{aligned} \delta \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q(P_\eta \mathcal{A}_\eta^R) \rangle &= \left\langle \frac{\partial}{\partial t} \{ \xi Y \delta(P_\eta \mathcal{A}_\eta^R) \}, Q(P_\eta \mathcal{A}_\eta^R) \right\rangle + \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q \delta(P_\eta \mathcal{A}_\eta^R) \rangle \\ &= \frac{\partial}{\partial t} \langle \xi Y \delta(P_\eta \mathcal{A}_\eta^R), Q(P_\eta \mathcal{A}_\eta^R) \rangle \end{aligned}$$

since using (2.8),  $\xi Q - X = -Q\xi$ , and  $\langle \partial_t (P_\eta \mathcal{A}_\eta^R), \delta(P_\eta \mathcal{A}_\eta^R) \rangle = 0$ , the following relation holds

$$\begin{aligned} \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q \delta(P_\eta \mathcal{A}_\eta^R) \rangle &= \langle (\xi Q - X) Y \partial_t (P_\eta \mathcal{A}_\eta^R), XY \delta(P_\eta \mathcal{A}_\eta^R) \rangle \\ &= \langle Q \partial_t (P_\eta \mathcal{A}_\eta^R), \xi Y \delta(P_\eta \mathcal{A}_\eta^R) \rangle \\ &= \langle \xi Y \delta(P_\eta \mathcal{A}_\eta^R), \frac{\partial}{\partial t} \{ Q(P_\eta \mathcal{A}_\eta^R) \} \rangle. \end{aligned}$$

Note however that the variation of the second ingredient provides an extra term

$$\begin{aligned} \delta \langle \partial_t (P_\eta \mathcal{A}_\eta^R), \mathcal{A}_\eta^R \rangle &= \left\langle \frac{\partial}{\partial t} \{ \delta(P_\eta \mathcal{A}_\eta^R) \}, \mathcal{A}_\eta^R \right\rangle + \langle \partial_t (P_\eta \mathcal{A}_\eta^R), \delta \mathcal{A}_\eta^R \rangle \\ &= \frac{\partial}{\partial t} \langle \delta(P_\eta \mathcal{A}_\eta^R), \mathcal{A}_\eta^R \rangle - \langle \delta(P_\eta \mathcal{A}_\eta^R), \partial_t \mathcal{A}_\eta^R \rangle + \langle \partial_t (P_\eta \mathcal{A}_\eta^R), \delta \mathcal{A}_\eta^R \rangle \\ &= \frac{\partial}{\partial t} \langle \delta(P_\eta \mathcal{A}_\eta^R), \mathcal{A}_\eta^R \rangle - \langle \delta \mathcal{A}_\eta^R, \partial_t \mathcal{A}_\eta^R \rangle. \end{aligned}$$

Here, we used  $P_\eta + P_\xi = 1$  and  $\langle \partial_t (P_\eta \mathcal{A}_\eta^R), \delta \mathcal{A}_\eta^R \rangle = \langle \partial_t \mathcal{A}_\eta^R, \delta(P_\xi \mathcal{A}_\eta^R) \rangle = - \langle \delta(P_\xi \mathcal{A}_\eta^R), \partial_t \mathcal{A}_\eta^R \rangle$ . As a result, the variation of the first term of  $S^R$  is given by

$$\delta \int_0^1 dt \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle = \langle \xi Y \delta(P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle + \int_0^1 dt \langle \delta \mathcal{A}_\eta^R, \partial_t \mathcal{A}_\eta^R \rangle. \quad (2.11)$$

Second, we compute the variation of the second term of  $S^R$ . Using (2.3), (2.7) for  $d = \partial_t, \delta$ , and Jacobi identities of the commutator, we obtain

$$\begin{aligned}
\delta \langle \mathcal{A}_t^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle &= \langle \delta \mathcal{A}_t^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle + \langle \mathcal{A}_t^{\text{NS}}, [\mathcal{A}_\eta^R, \delta \mathcal{A}_\eta^R] \rangle \\
&= \langle \partial \mathcal{A}_\delta^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle + \langle [\mathcal{A}_t^{\text{NS}}, \mathcal{A}_\delta^{\text{NS}}], m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle + \langle \mathcal{A}_t^{\text{NS}}, [\mathcal{A}_\eta^R, \delta \mathcal{A}_\eta^R] \rangle \\
&= \frac{\partial}{\partial t} \langle \mathcal{A}_\delta^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle - \langle \mathcal{A}_\delta^{\text{NS}}, [\mathcal{A}_\eta^R, \partial_t \mathcal{A}_\eta^R] \rangle \\
&\quad - \langle [\mathcal{A}_\delta^{\text{NS}}, \mathcal{A}_t^{\text{NS}}], m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle + \langle \delta \mathcal{A}_\eta^R, [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle \\
&= \frac{\partial}{\partial t} \langle \mathcal{A}_\delta^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle - \frac{1}{2} \langle [\mathcal{A}_\delta^{\text{NS}}, \mathcal{A}_t^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_\eta^R] \rangle \\
&\quad + \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], \partial_t \mathcal{A}_\eta^R \rangle + \langle \delta \mathcal{A}_\eta^R, [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle \\
&= \frac{\partial}{\partial t} \langle \mathcal{A}_\delta^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle - \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle \\
&\quad + \left( \langle D_\eta^{\text{NS}} \mathcal{A}_\delta^R, [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle + \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], D_\eta^{\text{NS}} \mathcal{A}_t^R \rangle \right)
\end{aligned}$$

In particular, from the forth line to the last line, we applied

$$\begin{aligned}
\langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], \partial_t \mathcal{A}_\eta^R \rangle &= \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], D_\eta^{\text{NS}} \mathcal{A}_t^R - [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle \\
&= \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], D_\eta^{\text{NS}} \mathcal{A}_t^R \rangle - \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle. \\
\langle \delta \mathcal{A}_\eta^R, [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle &= \langle D_\eta^{\text{NS}} \mathcal{A}_\delta^R - [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle \\
&= \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], D_\eta^{\text{NS}} \mathcal{A}_t^R \rangle - \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle. \\
-\frac{1}{2} \langle [\mathcal{A}_\delta^{\text{NS}}, \mathcal{A}_t^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_\eta^R] \rangle &= \frac{1}{2} \langle \mathcal{A}_\delta^{\text{NS}}, [[\mathcal{A}_\eta^R, \mathcal{A}_\eta^R], \mathcal{A}_t^{\text{NS}}] \rangle = -\langle \mathcal{A}_\delta^{\text{NS}}, [[\mathcal{A}_\eta^R, \mathcal{A}_t^R], \mathcal{A}_\eta^{\text{NS}}] \rangle \\
&= \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle.
\end{aligned}$$

As a result, the variation of the second term of  $S^R$  is given by

$$\delta \int_0^1 dt \langle \mathcal{A}_t^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle = \langle \mathcal{A}_\delta^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle - \int_0^1 dt \langle \delta \mathcal{A}_\eta^R, \partial_t \mathcal{A}_\eta^R \rangle \quad (2.12)$$

with the following relation

$$\begin{aligned}
\langle \delta \mathcal{A}_\eta^R, \partial_t \mathcal{A}_\eta^R \rangle &= \langle D_\eta^{\text{NS}} \mathcal{A}_\delta^R - [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], D_\eta^{\text{NS}} \mathcal{A}_t^R - [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle \\
&= \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle - \left( \langle D_\eta^{\text{NS}} \mathcal{A}_\delta^R, [\mathcal{A}_\eta^R, \mathcal{A}_t^{\text{NS}}] \rangle + \langle [\mathcal{A}_\eta^R, \mathcal{A}_\delta^{\text{NS}}], D_\eta^{\text{NS}} \mathcal{A}_t^R \rangle \right).
\end{aligned}$$

Hence, (2.11) plus (2.12) provides that  $S^R$  has topological  $t$ -dependence (2.10).

## 2.2 WZW-like complete action

We propose a Wess-Zumino-Witten complete action and show its gauge invariance on the basis of WZW-like relations (2.1 - 2.3) and (2.6 - 2.8).

### Action and equations of motion

We propose that a Wess-Zumino-Witten-like complete action is given by

$$\begin{aligned}
S_{\text{WZW}} &\equiv S^{\text{NS}} + S^R \\
&= \int_0^1 dt \left( \langle \xi Y \partial_t (P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle + \langle \mathcal{A}_t^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle \right). \quad (2.13)
\end{aligned}$$

Since  $S^{\text{NS}}$  and  $S^{\text{R}}$  have topological  $t$ -dependence, the variation of the action  $S_{\text{WZW}}$  is given by

$$\delta S_{\text{WZW}} = \langle \xi Y \delta(P_\eta \mathcal{A}_\eta^{\text{R}}), Q \mathcal{A}_\eta^{\text{R}} \rangle + \langle \mathcal{A}_\delta^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^{\text{R}}, \mathcal{A}_\eta^{\text{R}}) \rangle. \quad (2.14)$$

We therefore obtain the equations of motion

$$\text{NS} : \quad Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^{\text{R}}, \mathcal{A}_\eta^{\text{R}}) = 0, \quad (2.15)$$

$$\text{R} : \quad P_\eta(Q \mathcal{A}_\eta^{\text{R}}) = 0. \quad (2.16)$$

Let  $\Lambda^{\text{NS}}$ ,  $\Lambda^{\text{R}}$ , and  $\Omega^{\text{NS}}$  be NS, R, and NS gauge parameter fields which have ghost-and-picture number  $(-1|0)$ ,  $(-1|\frac{1}{2})$ , and  $(-1|1)$ , respectively. These  $\Lambda^{\text{NS}}$ ,  $\Lambda^{\text{R}}$ , and  $\Omega^{\text{NS}}$  all belong to the large Hilbert space. The action is invariant under two types of gauge transformations: the gauge transformations parametrized by  $\Lambda = \Lambda^{\text{NS}} + \Lambda^{\text{R}}$

$$\mathcal{A}_{\delta_\Lambda}^{\text{NS}} = Q \Lambda^{\text{NS}} + \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{R}} \rrbracket, \quad (2.17)$$

$$\delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}) = -P_\eta Q \left( \eta \Lambda^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \Lambda^{\text{R}} \rrbracket - \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{NS}} \rrbracket \right), \quad (2.18)$$

and the gauge transformations parametrized by  $\Omega = \Omega^{\text{NS}}$

$$\mathcal{A}_{\delta_\Omega}^{\text{NS}} = \eta \Omega^{\text{NS}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \Omega^{\text{NS}} \rrbracket, \quad \delta_\Omega(P_\eta \mathcal{A}_\eta^{\text{R}}) = 0. \quad (2.19)$$

### $\Lambda$ -gauge invariance

The  $\Lambda$ -gauge transformations of the action is given by

$$\delta_\Lambda S_{\text{WZW}} = \langle \xi Y \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), Q \mathcal{A}_\eta^{\text{R}} \rangle + \langle \mathcal{A}_{\delta_\Lambda}^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^{\text{R}}, \mathcal{A}_\eta^{\text{R}}) \rangle. \quad (2.20)$$

We show that  $\delta_\Lambda S_{\text{WZW}} = 0$  with  $\Lambda$ -gauge transformations of fields

$$\begin{aligned} \mathcal{A}_{\delta_\Lambda}^{\text{NS}} &= Q \Lambda^{\text{NS}} + \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{R}} \rrbracket, \\ \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}) &= -P_\eta Q \left( \eta \Lambda^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \Lambda^{\text{R}} \rrbracket - \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{NS}} \rrbracket \right), \end{aligned}$$

where  $\Lambda^{\text{NS}}$  is an NS gauge parameter carrying ghost-and-picture number  $(-1|0)$  and  $\Lambda^{\text{R}}$  is a R gauge parameter carrying ghost-and-picture number  $(-1|\frac{1}{2})$ . Note that these  $\Lambda^{\text{NS}}$  and  $\Lambda^{\text{R}}$  belong to the large Hilbert space  $\mathcal{H}$  but  $\delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}})$  has to be in the restricted one  $\mathcal{H}_R$ .

First, we consider the first term of (2.20) with (2.18). This term consists of two ingredients

$$\begin{aligned} \langle \xi Y \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), Q \mathcal{A}_\eta^{\text{R}} \rangle &= \langle \xi Y \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), Q(P_\eta + P_\xi) \mathcal{A}_\eta^{\text{R}} \rangle \\ &= \langle \xi Y \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), QXY(P_\eta \mathcal{A}_\eta^{\text{R}}) \rangle - \langle \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), P_\xi \mathcal{A}_\eta^{\text{R}} \rangle \\ &= \langle \xi Q \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), Y(P_\eta \mathcal{A}_\eta^{\text{R}}) \rangle - \langle \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), P_\xi \mathcal{A}_\eta^{\text{R}} \rangle. \end{aligned} \quad (2.21)$$

Here, we used  $\delta(P_\eta \mathcal{A}_\eta^{\text{R}}) = P_\eta(\delta P_\eta \mathcal{A}_\eta^{\text{R}})$ ,  $XY(\delta P_\eta \mathcal{A}_\eta^{\text{R}}) = \delta(XY P_\eta \mathcal{A}_\eta^{\text{R}})$ , and  $XY(P_\eta \mathcal{A}_\eta^{\text{R}}) = P_\eta \mathcal{A}_\eta^{\text{R}}$ . Since the first ingredient of (2.21) with  $\Lambda$ -gauge transformations (2.18) becomes

$$\begin{aligned} \langle \xi Q \delta_\Lambda(P_\eta \mathcal{A}_\eta^{\text{R}}), Y(P_\eta \mathcal{A}_\eta^{\text{R}}) \rangle &= -\langle \xi Q P_\eta Q(D_\eta^{\text{NS}} \Lambda^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{NS}} \rrbracket), Y(P_\eta \mathcal{A}_\eta^{\text{R}}) \rangle \\ &= \langle P_\xi Q \xi Q(D_\eta^{\text{NS}} \Lambda^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{NS}} \rrbracket), Y(P_\eta \mathcal{A}_\eta^{\text{R}}) \rangle \\ &= \langle Q(D_\eta^{\text{NS}} \Lambda^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{R}}, \Lambda^{\text{NS}} \rrbracket), P_\eta \mathcal{A}_\eta^{\text{R}} \rangle \end{aligned}$$

and the second ingredient of (2.21) with  $\Lambda$ -gauge transformations (2.18) becomes

$$\begin{aligned} -\langle \delta_\Lambda(P_\eta \mathcal{A}_\eta^R), P_\xi \mathcal{A}_\eta^R \rangle &= \langle P_\eta Q(D_\eta^{\text{NS}} \Lambda^R - \llbracket \mathcal{A}_\eta^R, \Lambda^{\text{NS}} \rrbracket), P_\xi \mathcal{A}_\eta^R \rangle \\ &= \langle Q(D_\eta^{\text{NS}} \Lambda^R - \llbracket \mathcal{A}_\eta^R, \Lambda^{\text{NS}} \rrbracket), P_\xi \mathcal{A}_\eta^R \rangle, \end{aligned}$$

we obtain

$$\begin{aligned} \langle \xi Y \delta_\Lambda(P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle &= \langle \xi Q \delta_\Lambda(P_\eta \mathcal{A}_\eta^R), Y(P_\eta \mathcal{A}_\eta^R) \rangle - \langle \delta_\Lambda(P_\eta \mathcal{A}_\eta^R), P_\xi \mathcal{A}_\eta^R \rangle \\ &= \langle Q(D_\eta^{\text{NS}} \Lambda^R - \llbracket \mathcal{A}_\eta^R, \Lambda^{\text{NS}} \rrbracket), P_\eta \mathcal{A}_\eta^R + P_\xi \mathcal{A}_\eta^R \rangle \\ &= -\langle D_\eta^{\text{NS}} \Lambda^R - \llbracket \mathcal{A}_\eta^R, \Lambda^{\text{NS}} \rrbracket, Q \mathcal{A}_\eta^R \rangle. \end{aligned} \quad (2.22)$$

Next, we compute the second term of (2.20) with (2.17). Using  $Q^2 = 0$ ,  $\llbracket \mathcal{A}_\eta^R, \mathcal{A}_\eta^R \rrbracket, \mathcal{A}_\eta^R \rrbracket = 0$ , and  $D_\eta^{\text{NS}} \mathcal{A}_\eta^R = 0$ , we quickly find that

$$\begin{aligned} \langle \mathcal{A}_{\delta_\Lambda}^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle &= \langle Q \Lambda^{\text{NS}} + \llbracket \mathcal{A}_\eta^R, \Lambda^R \rrbracket, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle \\ &= \langle \llbracket \mathcal{A}_\eta^R, \Lambda^R \rrbracket, Q \mathcal{A}_\eta^{\text{NS}} \rangle + \langle Q \Lambda^{\text{NS}}, m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle \\ &= -\langle \llbracket \mathcal{A}_\eta^R, \Lambda^R \rrbracket, D_\eta^{\text{NS}} \mathcal{A}_Q^{\text{NS}} \rangle - \langle \Lambda^{\text{NS}}, \llbracket \mathcal{A}_\eta^R, Q \mathcal{A}_\eta^R \rrbracket \rangle \\ &= \langle D_\eta^{\text{NS}} \Lambda^R, \llbracket \mathcal{A}_\eta^R, \mathcal{A}_Q^{\text{NS}} \rrbracket \rangle - \langle \llbracket \mathcal{A}_\eta^R, \Lambda^{\text{NS}} \rrbracket, Q \mathcal{A}_\eta^R \rangle. \end{aligned}$$

The property (2.7) of the R pure-gauge-like field  $-Q \mathcal{A}_\eta^R = D_\eta^{\text{NS}} \mathcal{A}_Q^R - \llbracket \mathcal{A}_\eta^R, \mathcal{A}_Q^{\text{NS}} \rrbracket$  gives

$$D_\eta^{\text{NS}} \left( Q \mathcal{A}_\eta^R - \llbracket \mathcal{A}_\eta^R, \mathcal{A}_Q^{\text{NS}} \rrbracket \right) = 0.$$

Hence, we obtain

$$\langle \mathcal{A}_{\delta_\Lambda}^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle = \langle D_\eta^{\text{NS}} \Lambda^R - \llbracket \mathcal{A}_\eta^R, \Lambda^{\text{NS}} \rrbracket, Q \mathcal{A}_\eta^R \rangle, \quad (2.23)$$

which just cancels (2.22), and we conclude  $\delta_\Lambda S_{\text{WZW}} = (2.22) + (2.23) = 0$  with (2.17) and (2.18).

### $\Omega$ -gauge invariance

The  $\Omega$ -gauge transformation of the action  $S_{\text{WZW}}$  is given by

$$\delta_\Omega S_{\text{WZW}} = \langle \xi Y \delta_\Omega(P_\eta \mathcal{A}_\eta^R), Q \mathcal{A}_\eta^R \rangle + \langle \mathcal{A}_{\delta_\Omega}^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle. \quad (2.24)$$

One can show that  $\delta_\Omega S_{\text{WZW}} = 0$  with  $\Omega$ -gauge transformations of fields

$$\mathcal{A}_{\delta_\Omega}^{\text{NS}} = \eta \Omega^{\text{NS}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \Omega^{\text{NS}} \rrbracket - \llbracket \mathcal{A}_\eta^R, \Omega^R \rrbracket, \quad (2.25)$$

$$\delta_\Omega(P_\eta \mathcal{A}_\eta^R) = -P_\eta Q \left( \eta \Omega^R - \llbracket \mathcal{A}_\eta^{\text{NS}}, \Omega^R \rrbracket \right), \quad (2.26)$$

where  $\Omega^{\text{NS}}$  is an NS gauge parameter carrying ghost-and-picture number  $(-1|1)$ ,  $\Omega^R$  is a R gauge parameter carrying ghost-and-picture number  $(-1|\frac{1}{2})$ , and both  $\Omega^{\text{NS}}$  and  $\Omega^R$  belong to the large Hilbert space. Note, however, that since R gauge parameters  $\Omega^R$  and  $\Lambda^R$  have the same ghost-and-picture number  $(-1|\frac{1}{2})$ , we can not distinguish these two parameters. As a result,  $\Omega^R$ -gauge transformation is absorbed into  $\Lambda^R$ -gauge transformation (2.18) and  $\Omega$ -gauge transformations (2.25) and (2.26) reduces to (2.19):

$$\mathcal{A}_{\delta_\Omega}^{\text{NS}} = \eta \Omega^{\text{NS}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \Omega^{\text{NS}} \rrbracket, \quad \delta_\Omega(P_\eta \mathcal{A}_\eta^R) = 0.$$

Then, using  $Q \mathcal{A}_\eta^{\text{NS}} = -D_\eta^{\text{NS}} \mathcal{A}_Q^{\text{NS}}$  and  $D_\eta^{\text{NS}} \mathcal{A}_\eta^R = 0$ , we quickly find that

$$\delta_\Omega S_{\text{WZW}} = \langle D_\eta^{\text{NS}} \Omega^{\text{NS}}, Q \mathcal{A}_\eta^{\text{NS}} + m_2(\mathcal{A}_\eta^R, \mathcal{A}_\eta^R) \rangle = 0.$$

Therefore, the action  $S_{\text{WZW}}$  is invariant under  $\Omega$ -gauge transformations (2.19).

### 2.3 Unified notation

We introduce a notation which is useful to unify the results of NS and R sectors. Then, the concept of Ramond number projections proposed in [2] naturally appears. We say Ramond number of the  $k$ -product  $M_k$  is  $n$  when number of R inputs of  $M_k$  minus number of R output of  $M_k$  equals to  $n$ . The symbol  $M_k|_n$  denotes the  $k$ -product projected onto Ramond number  $n$ . For example, R number 0 and 2 projection of the star product  $m_2$  are

$$\begin{aligned}\langle \text{NS} + \text{R}, m_2|_0(\text{NS} + \text{R}, \text{NS} + \text{R}) \rangle &= \langle \text{NS}, m_2(\text{NS}, \text{NS}) \rangle + \langle \text{R}, m_2(\text{NS}, \text{R}) + m_2(\text{R}, \text{NS}) \rangle, \\ \langle \text{NS} + \text{R}, m_2|_2(\text{NS} + \text{R}, \text{NS} + \text{R}) \rangle &= \langle \text{NS}, m_2(\text{R}, \text{R}) \rangle.\end{aligned}$$

It is helpful to specify whether the (output) state  $\mathcal{A}$  is NS or R. We write  $\mathcal{A}|^{\text{NS}}$  for the NS (output) state and  $\mathcal{A}|^{\text{R}}$  for the R (output) state. For example, for the sum of NS and R states

$$(\text{NS} + \text{R})|^{\text{NS}} = \text{NS}, \quad (\text{NS} + \text{R})|^{\text{R}} = \text{R}.$$

Then, we can write as follows:

$$\begin{aligned}m_2(\text{NS} + \text{R}, \text{NS} + \text{R})|_0^{\text{NS}} &= m_2(\text{NS}, \text{NS}), \quad m_2(\text{NS} + \text{R}, \text{NS} + \text{R})|_0^{\text{R}} = \llbracket \text{NS}, \text{R} \rrbracket, \\ m_2(\text{NS} + \text{R}, \text{NS} + \text{R})|_2^{\text{NS}} &= m_2(\text{R}, \text{R}), \quad m_2(\text{NS} + \text{R}, \text{NS} + \text{R})|_2^{\text{R}} = 0.\end{aligned}$$

#### Pure-gauge-like fields and associated fields

We can introduce a pure-gauge-like field including both sector

$$\mathcal{A}_\eta \equiv \mathcal{A}_\eta^{\text{NS}} + \mathcal{A}_\eta^{\text{R}} \quad (2.27)$$

such that  $\mathcal{A}_\eta$  satisfies

$$D_\eta \mathcal{A}_\eta \equiv \eta \mathcal{A}_\eta - m_2|_0(\mathcal{A}_\eta, \mathcal{A}_\eta) = 0. \quad (2.28)$$

Note that NS and R out-puts of  $D_\eta \mathcal{A}_\eta = 0$  give

$$\begin{aligned}\text{NS} : \quad & (D_\eta \mathcal{A}_\eta)|^{\text{NS}} \equiv \eta \mathcal{A}_\eta^{\text{NS}} - m_2(\mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_\eta^{\text{NS}}) = 0, \\ \text{R} : \quad & (D_\eta \mathcal{A}_\eta)|^{\text{R}} \equiv \eta \mathcal{A}_\eta^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_\eta^{\text{R}} \rrbracket = 0,\end{aligned}$$

which are just the defining equations of NS and R pure-gauge-like fields (2.1) and (2.6) respectively. Similarly, we can also define an associated field of  $d$  including both sector

$$\mathcal{A}_d \equiv \mathcal{A}_d^{\text{NS}} + \mathcal{A}_d^{\text{R}} \quad (2.29)$$

such that  $\mathcal{A}_d$  satisfies

$$(-)^d d \mathcal{A}_\eta = D_\eta \mathcal{A}_d, \quad (2.30)$$

whose NS out-put  $((-)^d d \mathcal{A}_\eta = D_\eta \mathcal{A}_d)|^{\text{NS}}$  and R out-put  $((-)^d d \mathcal{A}_\eta = D_\eta \mathcal{A}_d)|^{\text{R}}$  just provide the defining equations of NS and R pure-gauge-like fields (2.1) and (2.6) respectively

$$\begin{aligned}\text{NS} : \quad & (-)^d d \mathcal{A}_\eta^{\text{NS}} = \eta \mathcal{A}_d^{\text{NS}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_d^{\text{NS}} \rrbracket, \\ \text{R} : \quad & (-)^d d \mathcal{A}_\eta^{\text{R}} = \eta \mathcal{A}_d^{\text{R}} - \llbracket \mathcal{A}_\eta^{\text{NS}}, \mathcal{A}_d^{\text{R}} \rrbracket - \llbracket \mathcal{A}_\eta^{\text{R}}, \mathcal{A}_d^{\text{NS}} \rrbracket.\end{aligned}$$

## Action and equations of motion

In this notation, our Wess-Zumino-Witten-like complete action is given by

$$S_{\text{WZW}} = \int_0^1 dt \langle \mathcal{A}_t^*, Q\mathcal{A}_\eta + m_2|_2(\mathcal{A}_\eta, \mathcal{A}_\eta) \rangle, \quad (2.31)$$

where the conjugate associated field  $\mathcal{A}_t^*$  is defined by

$$\mathcal{A}_t^* \equiv \mathcal{A}_t^{\text{NS}} + \xi Y \partial_t (P_\eta \mathcal{A}_\eta^{\text{R}}). \quad (2.32)$$

Note that the projection onto Ramond number 2 states implies  $m_2|_2(\mathcal{A}_\eta, \mathcal{A}_\eta) = m_2(\mathcal{A}_\eta^{\text{R}}, \mathcal{A}_\eta^{\text{R}})$  for  $\mathcal{A}_t^{\text{NS}}$  and  $m_2|_2(\mathcal{A}_\eta, \mathcal{A}_\eta) = 0$  for  $\xi Y \partial_t (P_\eta \mathcal{A}_\eta^{\text{R}})$ . Then, the variation of the action becomes  $\delta S_{\text{WZW}} = \langle \mathcal{A}_\delta^*, Q\mathcal{A}_\eta + m_2|_2(\mathcal{A}_\eta, \mathcal{A}_\eta) \rangle$  with  $\mathcal{A}_\delta^* \equiv \mathcal{A}_\delta^{\text{NS}} + \xi Y \delta (P_\eta \mathcal{A}_\eta^{\text{R}})$  and the equations of motion is given by

$$Q\mathcal{A}_\eta + m_2|_2(\mathcal{A}_\eta, \mathcal{A}_\eta) = 0, \quad (2.33)$$

which reproduces NS and R equations of motion (2.15) and (2.16) by NS and R out-puts projections respectively. When we consider another parametrization of the action and its relation to the parametrization given in section 1.1, this notation would be useful.

## 3 Another parametrization

We use the same notation as [2] in this section. Readers who are unfamiliar with  $A_\infty$  algebras or coalgebraic computations see, for example, [2–5, 11, 23, 24, 48] or other mathematical manuscripts [?, 33, 34]. In the work of [2], the on-shell conditions of superstring field theories are proposed. For open superstring field theory, it is given by

$$\pi_1(Q + m_2|_2) \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} = 0, \quad (3.1)$$

where  $\tilde{\Psi} = \tilde{\Psi}^{\text{NS}} + \tilde{\Psi}^{\text{R}}$  is an NS plus R string field,  $Q$  is the BRST operator, and  $m_2|_2$  denotes the star product  $m_2$  with R number 2 projection. Note that NS and R out-puts of (3.1) are given by

$$\text{NS : } \quad Q \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{NS}} + m_2 \left( \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{R}}, \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{R}} \right) = 0, \quad (3.2)$$

$$\text{R : } \quad Q \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{R}} = 0. \quad (3.3)$$

Note also that in general, the cohomomorphism  $\hat{\mathbf{G}}$  is constructed by the path-ordered exponential (with direction) of a gauge product  $\boldsymbol{\mu}(t)$ , a coderivation, as follows

$$\hat{\mathbf{G}} \equiv \mathcal{P} \exp \left[ \int_0^t dt' \boldsymbol{\mu}(t') \right]. \quad (3.4)$$

In this paper, we always use  $\hat{\mathbf{G}}$  given in [2]. In [2], the gauge product  $\boldsymbol{\mu}(t)$  consists of R number 0 projected objects. Therefore,  $\pi_1 \hat{\mathbf{G}}$  has at most one Ramond state in its in-puts.

### 3.1 Another parametrization of the WZW-like complete action

In this section, we define pure-gauge-like and associated fields parametrized by  $\tilde{\Psi} = \tilde{\Psi}^{\text{NS}} + \tilde{\Psi}^{\text{R}}$  and construct a gauge invariant action, whose equations of motion equals to (3.1), the Ramond equations of motion proposed in [2]. The proofs of required properties are in section 3.2.

#### Parametrization inspired by Ramond equations of motion

We can construct an NS pure-gauge-like field  $\tilde{A}_\eta^{\text{NS}}$  by

$$\tilde{A}_\eta^{\text{NS}} \equiv \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{NS} = \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}^{\text{NS}}}, \quad (3.5)$$

and a R pure-gauge-like field  $\tilde{A}_\eta^{\text{R}}$  by

$$\tilde{A}_\eta^{\text{R}} \equiv \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{R} = \pi_1 \hat{\mathbf{G}} \left( \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \otimes \tilde{\Psi}^{\text{R}} \otimes \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \right). \quad (3.6)$$

These pure-gauge-like fields are parametrized by NS and R string field  $\tilde{\Psi} = \tilde{\Psi}^{\text{NS}} + \tilde{\Psi}^{\text{R}}$ . While the NS string field  $\tilde{\Psi}$  is a Grassmann odd and ghost-and-picture number  $(1| - 1)$  state in the small Hilbert space  $\mathcal{H}_S$ , the R string field  $\tilde{\Psi}^{\text{R}}$  is a Grassmann odd and ghost-and-picture number  $(1| - \frac{1}{2})$  state in the restricted small Hilbert space  $\mathcal{H}_R$ . Hence,  $\tilde{\Psi}^{\text{NS}} \in \mathcal{H}_S$  and  $\tilde{\Psi}^{\text{R}} \in \mathcal{H}_R$  satisfy

$$\eta \tilde{\Psi}^{\text{NS}} = 0, \quad \eta \tilde{\Psi}^{\text{R}} = 0, \quad XY \tilde{\Psi}^{\text{R}} = \tilde{\Psi}^{\text{R}}.$$

Note that  $\tilde{A}_\eta^{\text{NS}}$  has ghost-and-picture number  $(1| - 1)$  and  $\tilde{A}_\eta^{\text{R}}$  has ghost-and-picture number  $(1| - \frac{1}{2})$  by construction. As we will see in section 3.2, one can check that  $\tilde{A}_\eta^{\text{NS}}$  and  $\tilde{A}_\eta^{\text{R}}$  satisfy the defining properties of pure-gauge-like fields:

$$\text{NS :} \quad \eta \tilde{A}_\eta^{\text{NS}} - m_2(\tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^{\text{NS}}) = 0, \quad (3.7)$$

$$\text{R :} \quad \eta \tilde{A}_\eta^{\text{R}} - m_2(\tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^{\text{R}}) - m_2(\tilde{A}_\eta^{\text{R}}, \tilde{A}_\eta^{\text{NS}}) = 0. \quad (3.8)$$

Let  $d$  be a derivation operator commuting with  $\eta$ . Then, with these pure-gauge-like fields parametrized by small Hilbert space string fields  $\tilde{\Psi}$ , an NS associated field  $\tilde{A}_d^{\text{NS}}$  defined by

$$\tilde{A}_d^{\text{NS}} \equiv \pi_1 \hat{\mathbf{G}} \left( \xi_d \frac{1}{1 - \tilde{\Psi}} \right) \Big|^\text{NS} \quad (3.9)$$

and a R associated field  $\tilde{A}_d^{\text{R}}$  defined by

$$\tilde{A}_d^{\text{R}} \equiv \pi_1 \hat{\mathbf{G}} \left( \xi_d \frac{1}{1 - \tilde{\Psi}} \right) \Big|^\text{R} \quad (3.10)$$

satisfy the defining properties of associated fields:

$$(-)^d d \tilde{A}_\eta^{\text{NS}} = \eta \tilde{A}_d^{\text{NS}} - \llbracket \tilde{A}_\eta^{\text{NS}}, \tilde{A}_d^{\text{NS}} \rrbracket, \quad (3.11)$$

$$(-)^d d \tilde{A}_\eta^{\text{R}} = \eta \tilde{A}_d^{\text{R}} - \llbracket \tilde{A}_\eta^{\text{NS}}, \tilde{A}_d^{\text{R}} \rrbracket - \llbracket \tilde{A}_\eta^{\text{R}}, \tilde{A}_d^{\text{NS}} \rrbracket, \quad (3.12)$$

which we prove in section 3.2. Once the defining properties (3.7), (3.8), (3.11), and (3.12) are proved using pure-gauge-like fields  $\tilde{A}_\eta^{\text{NS}}$ ,  $\tilde{A}_\eta^{\text{R}}$  defined by (3.5), (3.6) and associated fields

$\tilde{A}_d^{\text{NS}}$ ,  $\tilde{A}_d^{\text{R}}$  defined by (3.9), (3.10), we can construct a gauge invariant action on the basis of Wess-Zumino-Witten-like framework proposed in section 2.

### Consistency with the XY-projection

To apply our WZW-like framework, we need the XY-projection invariance of  $P_\eta \tilde{A}_\eta^{\text{R}}$

$$XY(P_\eta \tilde{A}_\eta^{\text{R}}) = P_\eta \tilde{A}_\eta^{\text{R}}. \quad (3.13)$$

Unfortunately, for any choice of cohomomorphism  $\hat{\mathbf{G}}$ , the R pure-gauge-like field  $\tilde{A}_\eta^{\text{R}}$  defined by (3.6) does not satisfy this property. Note, however, that if we can take  $\hat{\mathbf{G}}$  satisfying

$$\xi \hat{\mathbf{G}} = \xi, \quad (3.14)$$

then the R pure-gauge-like field defined by (3.6) automatically satisfy (3.13) because

$$\begin{aligned} XY(P_\eta \tilde{A}_\eta^{\text{R}}) &= XY P_\eta \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{R} = XY \eta \pi_1 \xi \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{R} \\ &= XY P_\eta \pi_1 \frac{1}{1 - \tilde{\Psi}} \Big|^\text{R} = XY P_\eta \tilde{\Psi}^{\text{R}} = P_\eta \tilde{\Psi}^{\text{R}} = P_\eta \tilde{A}_\eta^{\text{R}}, \end{aligned}$$

with  $XY \tilde{\Psi}^{\text{R}} = \tilde{\Psi}^{\text{R}}$  and  $P_\eta = \eta \xi$  where  $\xi = \xi_0$  for NS states and  $\xi = \Xi$  for R states. Recall that  $\hat{\mathbf{G}}$  is constructed by the path-ordered exponential of a gauge product  $\boldsymbol{\mu}(t)$  as (3.4). When we take this gauge product as  $\xi$ -exact one  $\boldsymbol{\mu}(t) \equiv \xi \mathbf{M}(t)$ , the cohomomorphism  $\hat{\mathbf{G}}$  is given by

$$\hat{\mathbf{G}} \equiv \mathcal{P} e^{\int dt \xi \mathbf{M}(t)} = \mathbb{I} + \xi \left( M_2 + \frac{1}{2} M_3 + \frac{1}{2} M_2 \xi M_2 + \dots \right),$$

and it satisfies (3.14). This  $\xi$ -exact choice of the gauge product is always possible by using ambiguities of the construction of (intermediate) gauge products  $\boldsymbol{\mu}(t)$  or setting the initial condition of the defining equations of the  $A_\infty$  products in [2]. Note that although a naive choice of  $\xi$ -exact gauge products  $\boldsymbol{\mu} = \xi \mathbf{M}$  breaks the ciclic property of the  $A_\infty$  products  $\tilde{\mathbf{M}} = \hat{\mathbf{G}}^{-1} (\mathbf{Q} + \mathbf{m}_2|_2) \hat{\mathbf{G}}$ , it is no problem in our Wess-Zumino-Witten-like framework.

We would like to emphasize that it does not necessitate the ciclic property of  $\hat{\mathbf{G}}$  or  $\tilde{\mathbf{M}}$  to construct the Wess-Zumino-Witten-like complete action. We need the ciclic property of  $D_\eta$  only, which holds for any choice of  $\hat{\mathbf{G}}$ . Hence, we can always impose (3.13) in a consistent way with the definitions of pure-gauge-like fields (3.5), (3.6).

### Action and gauge invariance

Utilizing pure-gauge-like and associated fields satisfying (3.7), (3.8), (3.11), (3.12), and (3.13), we construct the Wess-Zumino-Witten-like complete action

$$\tilde{S} = \int_0^1 dt \left( \langle \xi Y \partial_t (P_\eta \tilde{A}_\eta^{\text{R}}), Q \tilde{A}_\eta^{\text{R}} \rangle + \langle \tilde{A}_t^{\text{NS}}, Q \tilde{A}_\eta^{\text{NS}} + m_2(\tilde{A}_\eta^{\text{R}}, \tilde{A}_\eta^{\text{R}}) \rangle \right), \quad (3.15)$$

which is parametrized by small Hilbert space string fields  $\tilde{\Psi} = \tilde{\Psi}^{\text{NS}} + \tilde{\Psi}^{\text{R}}$ . As we found in section 2, the variation of the action is given by

$$\delta \tilde{S} = \langle \xi Y \delta (P_\eta \tilde{A}_\eta^{\text{R}}), Q \tilde{A}_\eta^{\text{R}} \rangle + \langle \tilde{A}_\delta^{\text{NS}}, Q \tilde{A}_\eta^{\text{NS}} + m_2(\tilde{A}_\eta^{\text{R}}, \tilde{A}_\eta^{\text{R}}) \rangle, \quad (3.16)$$



and the action is invariant under two types of gauge transformations: the gauge transformations parametrized by  $\Lambda = \Lambda^{\text{NS}} + \Lambda^{\text{R}}$

$$\begin{aligned}\tilde{A}_{\delta\Lambda}^{\text{NS}} &= Q\Lambda^{\text{NS}} + \llbracket \tilde{A}_\eta^{\text{R}}, \Lambda^{\text{R}} \rrbracket, \\ \delta_\Lambda(P_\eta \tilde{A}_\eta^{\text{R}}) &= -P_\eta Q\left(\eta\Lambda^{\text{R}} - \llbracket \tilde{A}_\eta^{\text{NS}}, \Lambda^{\text{R}} \rrbracket - \llbracket \tilde{A}_\eta^{\text{R}}, \Lambda^{\text{NS}} \rrbracket\right),\end{aligned}$$

and the gauge transformations parametrized by  $\Omega = \Omega^{\text{NS}}$

$$\tilde{A}_{\delta\Omega}^{\text{NS}} = \eta\Omega^{\text{NS}} - \llbracket \tilde{A}_\eta^{\text{NS}}, \Omega^{\text{NS}} \rrbracket, \quad \delta_\Omega(P_\eta \tilde{A}_\eta^{\text{R}}) = 0.$$

Here NS, R, and NS gauge parameter fields  $\Lambda^{\text{NS}}$ ,  $\Lambda^{\text{R}}$ , and  $\Omega^{\text{NS}}$  have ghost-and-picture number  $(-1|0)$ ,  $(-1|\frac{1}{2})$ , and  $(-1|1)$ , respectively, and these fields all belong to the large Hilbert space. Note, however, that the gauge transformation  $\delta_\Lambda(P_\eta \tilde{A}_\eta^{\text{R}})$  has to be in the restricted small Hilbert space  $\mathcal{H}_R$ .

### Equations of motion

Since the action  $\tilde{S}$  has topological  $t$ -dependence and its variation is given by (3.16), we obtain the equations of motion

$$\text{NS : } \quad Q\tilde{A}_\eta^{\text{NS}} + m_2(\tilde{A}_\eta^{\text{R}}, \tilde{A}_\eta^{\text{R}}) = \pi_1(\mathbf{Q} + \mathbf{m}_2|_2) \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{NS}}^{\text{NS}} = 0, \quad (3.17)$$

$$\text{R : } \quad P_\eta(Q\tilde{A}_\eta^{\text{R}}) = P_\eta\left(\pi_1(\mathbf{Q} + \mathbf{m}_2|_2) \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}}\right) \Big|_{\text{R}}^{\text{R}} = 0, \quad (3.18)$$

which is equivalent to (3.1). While the NS out-put of the equations of motion (3.17) is the same as (3.2), the R out-put (3.18) is equal to the small Hilbert space component of (3.3). Note that  $P_\xi(Q\tilde{A}_\eta^{\text{R}})$  can not be determined from the action because it vanishes in the inner product and does not affect the value of the action. We thus set  $P_\xi(Q\tilde{A}_\eta^{\text{R}}) = 0$  and obtain (3.3).

### Kinetic term

It is interesting to compare kinetic terms of (3.15) and (1.1). In the present parametrization of (3.15), the kinetic term of  $\tilde{S}$  is given by

$$-\frac{1}{2}\langle \xi \tilde{\Psi}^{\text{NS}}, Q\tilde{\Psi}^{\text{NS}} \rangle - \frac{1}{2}\langle \xi Y \tilde{\Psi}^{\text{R}}, Q\tilde{\Psi}^{\text{R}} \rangle.$$

Note that the Ramond kinetic term is just equal to that of Kunitomo-Okawa's action. Similarly, we quickly check that the NS kinetic term is equivalent to that of Kunitomo-Okawa's action with the (trivial embedding) condition  $\tilde{\Psi}^{\text{NS}} = \eta\Phi^{\text{NS}}$  or the (linear partial gauge fixing) condition  $\Phi^{\text{NS}} = \xi\tilde{\Psi}^{\text{NS}}$ . Therefore, the kinetic term of  $\tilde{S}$  has the same spectrum as that of [1].

## 3.2 WZW-like relation from $A_\infty$ and $\eta$ -exactness

### Dual $A_\infty$ -products and Derivation properties

Let  $\eta$  be the coderivation constructed from  $\eta$ , which is nilpotent  $\eta^2 = 0$ , and let  $\mathbf{a}$  be a nilpotent coderivation satisfying  $\mathbf{a}\eta = -(-)^{\mathbf{a}\eta}\eta\mathbf{a}$  and  $\mathbf{a}^2 = 0$ . Then, we assume that  $\hat{\mathbf{G}}^{-1} : (\mathcal{H}, \mathbf{a}) \rightarrow (\mathcal{H}_S, \mathbf{D}_\mathbf{a})$  is an  $A_\infty$ -morphism, where  $\mathcal{H}$  is the large Hilbert space,  $\mathcal{H}_S$  is the small

Hilbert space, and  $D_a \equiv \hat{G}^{-1} a \hat{G}$ . Note that  $D_a$  is nilpotent:  $D_a^2 = (\hat{G}^{-1} a \hat{G})(\hat{G}^{-1} a \hat{G}) = \hat{G}^{-1} a^2 \hat{G} = 0$ . For example, one can use  $Q$ ,  $Q + m_2|_2$ , and so on for  $a$ , and various  $\hat{G}$  appearing in [2, 23, 24] for  $\hat{G}$ . Suppose that the coderivation  $D_a$  also commutes with  $\eta$ , which means

$$(D_a)^2 = 0, \quad \llbracket D_a, \eta \rrbracket = 0. \quad (3.19)$$

Then, we can introduce a dual  $A_\infty$ -products  $D^\eta$  defined by

$$D^\eta \equiv \hat{G} \eta \hat{G}^{-1}. \quad (3.20)$$

Note that the pair of nilpotent maps  $(D^\eta, a)$  have the same properties as  $(D_a, \eta)$ :

$$(D^\eta)^2 = 0, \quad \llbracket D^\eta, a \rrbracket = 0. \quad (3.21)$$

We can quickly find when the  $A_\infty$ -products  $D_a$  commutes with the coderivation  $\eta$  as (3.19), its dual  $A_\infty$ -product  $D^\eta$  and coderivation  $a$  also satisfies (3.21) as follows

$$\begin{aligned} a D^\eta &= (\hat{G} \hat{G}^{-1}) a (\hat{G} \eta \hat{G}^{-1}) = \hat{G} D_a \eta \hat{G}^{-1} \\ &= (-)^{a\eta} \hat{G} \eta D_a \hat{G}^{-1} = (-)^{a\eta} \hat{G} \eta \hat{G}^{-1} a \hat{G} \hat{G}^{-1} = (-)^{a\eta} D^\eta a. \end{aligned}$$

In this paper, as these coderivation  $a$  and  $A_\infty$ -morphism  $\hat{G}$ , we always use  $a \equiv Q + m_2|_2$  and  $\hat{G}$  introduced in (3.14), namely, a gauge product  $\hat{G}$  given by [2] with the choice satisfying  $\xi \hat{G} = \xi$ . Therefore, the dual  $A_\infty$  products is always given by

$$D^\eta = \eta - m_2|_0, \quad (3.22)$$

and the symbol  $D^\eta$  always denotes (3.22) in the rest. (See section 6.2 of [2].) Then, the Maurer-Cartan element of  $D^\eta = \eta - m_2|_0$  is given by

$$\begin{aligned} D^\eta \frac{1}{1 - \tilde{A}} &= \frac{1}{1 - \tilde{A}} \otimes \pi_1 \left( D^\eta \frac{1}{1 - \tilde{A}} \right) \otimes \frac{1}{1 - \tilde{A}} \\ &= \frac{1}{1 - \tilde{A}} \otimes \left( \eta \tilde{A} - m_2|_0(\tilde{A}, \tilde{A}) \right) \otimes \frac{1}{1 - \tilde{A}}, \end{aligned}$$

where  $\tilde{A} = \tilde{A}^{\text{NS}} + \tilde{A}^{\text{R}}$  is a state of the large Hilbert space  $\mathcal{H}$  and  $\pi_1$  is an natural 1-state projection onto  $\mathcal{H}$ . Hence, the solution of the Maurer-Cartan equation  $D^\eta(1 - \tilde{A})^{-1} = 0$  is given by a state  $\tilde{A}_\eta = \tilde{A}_\eta^{\text{NS}} + \tilde{A}_\eta^{\text{R}}$  satisfying

$$\eta \tilde{A}_\eta - m_2|_0(\tilde{A}_\eta, \tilde{A}_\eta) = 0, \quad (3.23)$$

and vice versa, the solution  $\tilde{A}_\eta = \tilde{A}_\eta^{\text{NS}} + \tilde{A}_\eta^{\text{R}}$  satisfies (3.23), or equivalently,

$$\begin{aligned} \text{NS :} \quad & \eta \tilde{A}_\eta^{\text{NS}} - m_2(\tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^{\text{NS}}) = 0, \\ \text{R :} \quad & \eta \tilde{A}_\eta^{\text{R}} - m_2(\tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^{\text{R}}) - m_2(\tilde{A}_\eta^{\text{R}}, \tilde{A}_\eta^{\text{NS}}) = 0, \end{aligned}$$

which is just equivalent to the condition (3.7) and (3.8) characterizing NS and R pure-gauge-like fields  $\tilde{A}_\eta^{\text{NS}}$  and  $\tilde{A}_\eta^{\text{R}}$ . As a result, we obtain one of the most important fact the solutions of the Maurer-Cartan equation of  $D^\eta = \eta - m_2|_0$  gives desired NS and R pure-gauge-like fields.

## NS and R pure-gauge-like fields

When the  $\boldsymbol{\eta}$ -complex  $(\mathcal{H}, \boldsymbol{\eta})$  is exact, there exist  $\boldsymbol{\xi}$  such that  $[\boldsymbol{\eta}, \boldsymbol{\xi}] = \mathbf{1}$  and  $\mathcal{H}$ , the large Hilbert space, is decomposed into the direct sum of  $\eta$ -exacts and  $\xi$ -exacts  $\mathcal{H} = P_\eta \mathcal{H} \oplus P_\xi \mathcal{H}$ , where  $P_\eta$  and  $P_\xi$  are projector onto  $\eta$ -exact and  $\xi$ -exact states respectively.<sup>3</sup> Note that since the small Hilbert space  $\mathcal{H}_S$  is defined by  $\mathcal{H}_S \equiv P_\eta \mathcal{H}$  and satisfies  $\mathcal{H}_S \subset P_\eta \mathcal{H}_S$ , all the states  $\tilde{\Psi}$  belonging to  $\mathcal{H}_S$  satisfy  $P_\eta \tilde{\Psi} = \tilde{\Psi}$  and  $P_\xi \tilde{\Psi} = 0$ , or simply,

$$\boldsymbol{\eta} \Psi = 0.$$

Using this fact, we can construct desired pure-gauge-like fields  $\tilde{A}_\eta^{\text{NS}}$  and  $\tilde{A}_\eta^{\text{R}}$  as solutions of the Maurer-Cartan equation of  $\boldsymbol{D}^\eta = \boldsymbol{\eta} - \boldsymbol{m}_2|_0$ . Note that the Maurer-Cartan equation consists of NS and R out-puts

$$\begin{aligned} \text{NS : } \quad \pi_1 \boldsymbol{D}^\eta \frac{1}{1 - \tilde{A}_\eta} \Big|^\text{NS} &= \pi_1 \boldsymbol{D}^\eta \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} = 0, \\ \text{R : } \quad \pi_1 \boldsymbol{D}^\eta \frac{1}{1 - \tilde{A}_\eta} \Big|^\text{R} &= \pi_1 \boldsymbol{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} \otimes \tilde{A}_\eta^{\text{R}} \otimes \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} \right) = 0, \end{aligned}$$

where the upper index of  $|$  denotes the NS or R projection: for any state  $\tilde{A} = \tilde{A}^{\text{NS}} + \tilde{A}^{\text{R}} \in \mathcal{H}$ , the NS projection  $\tilde{A}|^\text{NS}$  is defined by  $\tilde{A}|^\text{NS} \equiv \tilde{A}^{\text{NS}}$  and the R projection  $\tilde{A}|^\text{R}$  is defined by  $\tilde{A}|^\text{R} \equiv \tilde{A}^{\text{R}}$ .

An NS pure-gauge-like field  $\tilde{A}_\eta^{\text{NS}}$  is given by

$$\tilde{A}_\eta^{\text{NS}} \equiv \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{NS} = \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}^{\text{NS}}}$$

because it becomes a trivial NS state solution of the Maurer-Cartan equation as follows

$$\begin{aligned} \boldsymbol{D}^\eta \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} &= \boldsymbol{D}^\eta \frac{1}{1 - \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}^{\text{NS}}}} = \boldsymbol{D}^\eta \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} = \hat{\mathbf{G}} \boldsymbol{\eta} \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \\ &= \hat{\mathbf{G}} \left( \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \otimes \boldsymbol{\eta} \tilde{\Psi}^{\text{NS}} \otimes \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \right) = 0. \end{aligned}$$

Note that  $\pi_1 \boldsymbol{D}^\eta (1 - \tilde{A}_\eta^{\text{NS}})^{-1} = 0$  is equal to

$$\boldsymbol{\eta} \tilde{A}_\eta^{\text{NS}} - \boldsymbol{m}_2(\tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^{\text{NS}}) = 0.$$

Similarly, a R pure-gauge-like field  $\tilde{A}_\eta^{\text{R}}$  is given by

$$\tilde{A}_\eta^{\text{R}} \equiv \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{R} = \pi_1 \hat{\mathbf{G}} \left( \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \otimes \tilde{\Psi}^{\text{R}} \otimes \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \right)$$

because it becomes a trivial R state solution of the Maurer-Cartan equation as follows

$$\begin{aligned} \boldsymbol{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} \otimes \tilde{A}_\eta^{\text{R}} \otimes \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} \right) &= \boldsymbol{D}^\eta \hat{\mathbf{G}} \left( \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \otimes \tilde{\Psi}^{\text{R}} \otimes \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \right) \\ &= \hat{\mathbf{G}} \boldsymbol{\eta} \left( \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \otimes \tilde{\Psi}^{\text{R}} \otimes \frac{1}{1 - \tilde{\Psi}^{\text{NS}}} \right) = 0. \end{aligned}$$

---

<sup>3</sup>These satisfy  $P_\eta^2 = P_\eta$ ,  $P_\xi^2 = P_\xi$ ,  $P_\eta P_\xi = P_\xi P_\eta = 0$ , and  $P_\eta + P_\xi = 1$  on  $\mathcal{H}$ .

Hence, the R state solution  $\tilde{A}_\eta^R$  satisfies

$$\eta \tilde{A}_\eta^R - \llbracket \tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^R \rrbracket = 0.$$

Note that  $\llbracket \tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^R \rrbracket = m_2(\tilde{A}_\eta^{\text{NS}}, \tilde{A}_\eta^R) + m_2(\tilde{A}_\eta^R, \tilde{A}_\eta^{\text{NS}})$ .

#### Shift of the dual $A_\infty$ products $\mathbf{D}^\eta$

We introduce the  $\tilde{A}_\eta$ -shifted products  $[B_1, \dots, B_n]_{\tilde{A}_\eta}^\eta$  defined by

$$[B_1, \dots, B_n]_{\tilde{A}_\eta}^\eta \equiv \pi_1 \mathbf{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes B_1 \otimes \frac{1}{1 - \tilde{A}_\eta} \otimes \dots \otimes \frac{1}{1 - \tilde{A}_\eta} \otimes B_n \otimes \frac{1}{1 - \tilde{A}_\eta} \right). \quad (3.24)$$

Note that higher shifted products all vanish  $[B_1, \dots, B_{n>2}]_{\tilde{A}_\eta}^\eta = 0$  because now we consider  $\mathbf{D}^\eta \equiv \eta - m_2|_0$ . In particular, we write  $D_\eta B$  for  $[B]_\eta$ :

$$D_\eta B \equiv \pi_1 \mathbf{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes B \otimes \frac{1}{1 - \tilde{A}_\eta} \right), \quad (3.25)$$

or equivalently, for  $\tilde{A}_\eta = \tilde{A}_\eta^{\text{NS}} + \tilde{A}_\eta^R$  and  $B = B^{\text{NS}} + B^R$ ,

$$\begin{aligned} \text{NS : } \quad D_\eta B|^\text{NS} &= \eta B^{\text{NS}} - \llbracket \tilde{A}_\eta^{\text{NS}}, B^{\text{NS}} \rrbracket, \\ \text{R : } \quad D_\eta B|^\text{R} &= \eta B^R - \llbracket \tilde{A}_\eta^{\text{NS}}, B^R \rrbracket - \llbracket \tilde{A}_\eta^R, B^{\text{NS}} \rrbracket. \end{aligned}$$

When  $\tilde{A}_\eta$  gives a solution on the Maurer-Cartan equation of  $\mathbf{D}_\eta$ , these  $\tilde{A}_\eta$ -shifted products also satisfy  $A_\infty$ -relations, which implies that the linear operator  $D_\eta$  becomes nilpotent. We find

$$\begin{aligned} (D_\eta)^2 B &= \pi_1 \mathbf{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes \pi_1 \mathbf{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes B \otimes \frac{1}{1 - \tilde{A}_\eta} \right) \otimes \frac{1}{1 - \tilde{A}_\eta} \right) \\ &= \pi_1 (\mathbf{D}^\eta)^2 \frac{1}{1 - \tilde{A}_\eta} - \llbracket \pi_1 \mathbf{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \right), B \rrbracket_{\tilde{A}_\eta}^\eta = 0. \end{aligned}$$

#### NS and R associated fields

Let  $\mathbf{d}$  be a coderivation constructed from a derivation  $d$  of the dual  $A_\infty$ -products  $\mathbf{D}^\eta$ , which implies that the  $d$ -derivation property  $\llbracket \mathbf{d}, \mathbf{D}^\eta \rrbracket = 0$  holds. Then, we obtain  $\llbracket \mathbf{D}_d, \eta \rrbracket = 0$  with  $\mathbf{D}_d \equiv \hat{\mathbf{G}}^{-1} \mathbf{d} \hat{\mathbf{G}}$ , which means that  $\mathbf{D}_d$  is “ $\eta$ -exact” and there exists a coderivation  $\xi_d$  such that

$$\mathbf{D}_d = \hat{\mathbf{G}}^{-1} \mathbf{d} \hat{\mathbf{G}} = (-)^d \llbracket \eta, \xi_d \rrbracket. \quad (3.26)$$

Using this coderivation  $\xi_d$ , we can construct NS and R associated fields constructed from the derivation operator  $d$ . Note that the response of  $\mathbf{d}$  acting on the group-like element of  $\tilde{A}_\eta = \tilde{A}_\eta^{\text{NS}} + \tilde{A}_\eta^R$  is given by

$$\begin{aligned} (-)^d \mathbf{d} \frac{1}{1 - \tilde{A}_\eta} &= (-)^d G G^{-1} \mathbf{d} G \frac{1}{1 - \tilde{\Psi}} = G \eta \xi_d \frac{1}{1 - \tilde{\Phi}} = \mathbf{D}^\eta G \left( \xi_d \frac{1}{1 - \tilde{\Psi}} \right) \\ &= \mathbf{D}^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes \pi_1 G \left( \xi_d \frac{1}{1 - \tilde{\Psi}} \right) \otimes \frac{1}{1 - \tilde{A}_\eta} \right). \end{aligned} \quad (3.27)$$

An NS associated field of  $d$  is given by

$$\tilde{A}_d^{\text{NS}} \equiv \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \Big|^\text{NS}$$

because one can directly check

$$\begin{aligned} (-)^d d \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} &= (-)^d d \frac{1}{1 - \tilde{A}_\eta} \Big|^\text{NS} = D^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \otimes \frac{1}{1 - \tilde{A}_\eta} \right) \Big|^\text{NS} \\ &= D^\eta \left( \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} \otimes \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \Big|^\text{NS} \otimes \frac{1}{1 - \tilde{A}_\eta^{\text{NS}}} \right). \end{aligned}$$

Picking up the relation on  $\mathcal{H}$ , or equivalently acting  $\pi_1$  on this relation on  $T(\mathcal{H})$ , we obtain

$$(-)^d d \tilde{A}_\eta^{\text{NS}} = \eta \tilde{A}_d^{\text{NS}} - \llbracket \tilde{A}_\eta^{\text{NS}}, \tilde{A}_d^{\text{NS}} \rrbracket,$$

which is the simplest case of  $(-)^d d \tilde{A}_\eta = \pi_1 D^\eta$ -exact term.

Similarly, an R associated field of  $d$  is given by

$$\tilde{A}_d^{\text{R}} \equiv \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \Big|^\text{R}$$

because one can directly check

$$\begin{aligned} (-)^d d \tilde{A}_\eta^{\text{R}} &= \pi_1 (-)^d d \frac{1}{1 - \tilde{A}_\eta} \Big|^\text{R} = \pi_1 D^\eta \left( \frac{1}{1 - \tilde{A}_\eta} \otimes \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \otimes \frac{1}{1 - \tilde{A}_\eta} \right) \Big|^\text{R} \\ &= \eta \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \Big|^\text{R} - \llbracket \tilde{A}_\eta^{\text{NS}}, \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \Big|^\text{R} \rrbracket - \llbracket \tilde{A}_\eta^{\text{R}}, \pi_1 G\left(\xi_d \frac{1}{1 - \tilde{\Psi}}\right) \Big|^\text{NS} \rrbracket \\ &= \eta \tilde{A}_d^{\text{R}} - \llbracket \tilde{A}_\eta^{\text{NS}}, \tilde{A}_d^{\text{R}} \rrbracket - \llbracket \tilde{A}_\eta^{\text{R}}, \tilde{A}_d^{\text{NS}} \rrbracket. \end{aligned}$$

We obtain  $(-)^d d \tilde{A}_\eta^{\text{R}} = D_\eta^{\text{NS}} \tilde{A}_d^{\text{R}} + \llbracket \tilde{A}_\eta^{\text{R}}, \tilde{A}_d^{\text{NS}} \rrbracket$ , namely,  $(-)^d d \tilde{A}_\eta = \pi D^\eta$ -exact terms.

### 3.3 Equivalence of EKS and KO theories

In section 2, we proposed the Wess-Zumino-Witten-like complete action

$$S_{\text{WZW}} = \int_0^1 dt \langle \mathcal{A}_t^*, Q \mathcal{A}_\eta + m_2 |_2 (\mathcal{A}_\eta, \mathcal{A}_\eta) \rangle,$$

where  $\mathcal{A}_\eta \equiv \mathcal{A}_\eta^{\text{NS}} + \mathcal{A}_\eta^{\text{R}}$  and  $\mathcal{A}_t^* \equiv \mathcal{A}_t^{\text{NS}} + \xi Y \partial_t (P_\eta \mathcal{A}_\eta^{\text{R}})$ .

We found that one realization of this WZW-like complete action is given by setting

$$\mathcal{A}_\eta^{\text{NS}} := (\eta e^{\Phi^{\text{NS}}}) e^{-\Phi^{\text{NS}}} \equiv A_\eta^{\text{NS}}, \quad \mathcal{A}_\eta^{\text{R}} := F \Psi^{\text{R}} \equiv A_\eta^{\text{R}}, \quad (3.28)$$

which is just Kunitomo-Okawa's action proposed in [1]. Another realization of the action, which was proposed in section 3.1 and checked in section 3.2, is given by setting

$$\mathcal{A}_\eta^{\text{NS}} := \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{NS} \equiv \tilde{A}_\eta^{\text{NS}}, \quad \mathcal{A}_\eta^{\text{R}} := \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|^\text{R} \equiv \tilde{A}_\eta^{\text{R}}, \quad (3.29)$$

which reproduces the Ramond equations of motion proposed in [2]. Note also that the kinetic terms of (1.1) and (3.15) have the same spectrum. As a result, we obtain the equivalence of two theories proposed in [1] and [2], which are different parametrizations of (2.31). See also [3, 11].

In other words, since both (3.28) and (3.29) have the same WZW-like structure and gives the same WZW-like action (2.31), we can identify  $A_\eta = A_\eta^{\text{NS}} + A_\eta^{\text{R}}$  and  $\tilde{A}_\eta = \tilde{A}_\eta^{\text{NS}} + \tilde{A}_\eta^{\text{R}}$  in the same way as [3]. Then, the identification of pure-gauge-fields

$$A_\eta = (\eta e^{\Phi^{\text{NS}}})e^{-\Phi^{\text{NS}}} + F\Psi^{\text{R}} \cong \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{NS}} + \pi_1 \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \Big|_{\text{R}} = \tilde{A}_\eta \quad (3.30)$$

trivially provides the equivalence of two actions (1.1) and (3.15), namely the equivalence of two theories. Note that (3.30) gives the equivalence of two actions but does not *directly* give a field redefinition of two theories. A partial gauge fixing  $\Phi^{\text{NS}} = \xi \Psi^{\text{NS}}$  is necessitated to relate  $(\Psi^{\text{NS}}, \Psi^{\text{R}})$  and  $(\tilde{\Psi}^{\text{NS}}, \tilde{\Psi}^{\text{R}})$ . See also [4–6, 11]. Similarly, as demonstrated in [5], if we start with

$$\begin{aligned} A_t &= (\partial_t e^{\Phi^{\text{NS}}})e^{-\Phi^{\text{NS}}} + F\Xi \left( \partial_t \Psi^{\text{R}} - \llbracket (\partial_t e^{\Phi^{\text{NS}}})e^{-\Phi^{\text{NS}}}, F\Psi^{\text{R}} \rrbracket \right) \\ &\cong \pi_1 \hat{\mathbf{G}} \left( \xi_t \frac{1}{1 - \tilde{\Psi}} \right) \Big|_{\text{NS}} + \pi_1 \hat{\mathbf{G}} \left( \xi_t \frac{1}{1 - \tilde{\Psi}} \right) \Big|_{\text{R}} = \tilde{A}_t, \end{aligned} \quad (3.31)$$

we can read a field redefinition of  $(\Phi^{\text{NS}}, \Psi^{\text{R}})$  and  $(\tilde{\Phi}^{\text{NS}}, \tilde{\Psi}^{\text{R}})$  in the large Hilbert space for NS fields and in the restricted small space for R fields with a trivial up-lift  $\tilde{\Psi}^{\text{NS}} = \eta \tilde{\Phi}^{\text{NS}}$ . One can check that the same logic used for the NS sector in [5] also goes in the case including the R sector because WZW-like relations exist as we explained. See appendix A.

## 4 Conclusion

In this paper, we have clarified a Wess-Zumino-Witten-like structure including Ramond fields and proposed one systematic way to construct gauge invariant actions, which we call WZW-like complete action. In this framework, once a WZW-like functional  $\mathcal{A}_\eta = \mathcal{A}_\eta[\Psi]$  of some dynamical string field  $\Psi$  is constructed, one obtain one realization of our WZW-like complete action parametrized by  $\Psi$ . On the basis of this way, we have constructed an action  $\tilde{S}$  whose on-shell condition is equivalent to the Ramond equations of motion proposed in [2]. In particular, this action  $\tilde{S}$  and Kunitomo-Okawa's action proposed in [1] just give different parametrizations of the same WZW-like structure and action, which implies the equivalence of two theories [1, 2]. Let us conclude by discussing future directions.

### Closed superstring field theories

It would be interesting to extend the result of [1] to closed superstring field theories [49]. We expect that our idea of WZW-like structure and action also goes in heterotic and type II theories if the kinetic terms are given by the same form. Then, we need explicit expressions of  $D^\eta$  and  $\mathbf{l}$ , where  $D^\eta$  is a dual  $L_\infty$  structure of the original  $L_\infty$  products  $\tilde{\mathbf{L}} = \hat{\mathbf{G}}^{-1} \mathbf{l} \hat{\mathbf{G}}$  given in [2]. NS and NS-NS parts of these dual  $A_\infty/L_\infty$  structures are discussed in [11].

### Quantization and Supermoduli

We would have to quantize the (WZW-like) complete action and clarify its relation with supermoduli of super-Riemann surfaces [35–39] to obtain a better understanding of superstrings from recent developments in field theoretical approach. The Batalin-Vilkovisky formalism [43,44] is one helpful way to tackle these problems: A quantum master action is necessitated. As a first step, it is important to clarify whether we can obtain an  $A_\infty$ -morphism  $\widehat{\mathbf{G}}$  which has the cyclic property consistent with the  $XY$ -projection. If it is possible, the resultant action would have an  $A_\infty$  form and then the classical Batalin-Vilkovisky quantization is straightforward. A positive answer would be provided in upcoming work [50] for open superstring field theory without stubs. It would also be helpful to clarify more detailed relations between recent important developments.

## Acknowledgement

The author would like to thank Keiyu Goto, Hiroshi Kunitomo, and Yuji Okawa for comments. The author also would like to express his gratitude to his doctors, nurses, and all the staffs of University Hospital, Kyoto Prefectural University of Medicine, for medical treatments and care during his long hospitalization. This work was supported in part by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

## A Basic facts and some identities

We summarize important properties of the BPZ inner product and give proofs of some relations which we skipped in the text.

### BPZ properties

The BPZ inner product  $\langle A, B \rangle$  in the large Hilbert space of any  $A, B \in \mathcal{H}$  has the following properties with the BRST operator  $Q$  and the Witten's star product  $m_2$ :

$$\begin{aligned}\langle A, B \rangle &= (-)^{AB} \langle B, A \rangle, \\ \langle A, QB \rangle &= -(-)^A \langle QA, B \rangle, \\ \langle A, m_2(B, C) \rangle &= (-)^{A(B+C)} \langle B, m_2(C, A) \rangle.\end{aligned}$$

Note also that with a projector onto  $\eta$ -exact states  $P_\eta$  and  $P_\xi = 1 - P_\eta$  and the zero mode of  $\eta \equiv \eta_0$  of  $\eta(z)$ -current, the BPZ inner product satisfies

$$\begin{aligned}\langle P_\eta A, B \rangle &= \langle A, P_\xi B \rangle, \\ \langle A, \eta B \rangle &= (-)^A \langle \eta A, B \rangle.\end{aligned}$$

Similarly, for any states in the restricted small space  $A^R, B^R \in \mathcal{H}_R$ , the bilinear  $\langle \xi Y A^R, B^R \rangle$  has the following properties:

$$\begin{aligned}\langle \xi Y A^R, QB^R \rangle &= (-)^{AB} \langle \xi Y B^R, A^R \rangle, \\ \langle \xi Y A^R, QB^R \rangle &= -(-)^A \langle \xi Y QA^R, B^R \rangle.\end{aligned}$$

### Associated fields in Kunitomo-Okawa theory

In the work of [1], for any state  $B \in \mathcal{H}$ , the linear map  $F$  is defined by

$$FB \equiv \sum_{n=0}^{\infty} (\Xi \llbracket A_{\eta}^{\text{NS}}, \rrbracket)^n B = \frac{1}{1 - \Xi(\eta - D_{\eta}^{\text{NS}})} B,$$

where  $D_{\eta}^{\text{NS}} B = \eta B - \llbracket A_{\eta}^{\text{NS}}, B \rrbracket$  and  $(D_{\eta}^{\text{NS}})^2 = 0$ . Thus, its inverse is given by  $F^{-1} = \eta \Xi + \Xi D_{\eta}^{\text{NS}}$ , which provides  $\eta F^{-1} = F^{-1} D_{\eta}^{\text{NS}}$  or equivalently,  $F \eta F^{-1} = D_{\eta}^{\text{NS}}$ , and thus  $\llbracket D_{\eta}^{\text{NS}}, F \Xi \rrbracket = 1$ . Then, we find that  $A_{\eta}^{\text{R}} \equiv F \Psi$  satisfies  $D_{\eta}^{\text{NS}} A_{\eta}^{\text{R}} = 0$  as follows:

$$A_{\eta}^{\text{R}} \equiv F \Psi^{\text{R}} = F \eta \xi \Psi^{\text{R}} = D_{\eta}^{\text{NS}} F \Xi \Psi^{\text{R}} = D_{\eta}^{\text{NS}} F \Xi A_{\eta}^{\text{R}}.$$

With  $F^{-1} = \eta \Xi + \Xi D_{\eta}^{\text{NS}}$  and  $\llbracket d, \eta \rrbracket = 0$ , a derivation  $d$  acts on the state  $F \Psi$  as

$$\begin{aligned} d(F \Psi) &= F(d \Psi) - F \llbracket d, F^{-1} \rrbracket F \Psi \\ &= F(d \Psi) - F \llbracket d, \eta \Xi \rrbracket F \Psi - F \llbracket d, \Xi D_{\eta}^{\text{NS}} \rrbracket F \Psi \\ &= F(d \Psi) - (-)^d F(\eta \llbracket d, \Xi \rrbracket + (-)^d \llbracket d, \Xi \rrbracket D_{\eta}^{\text{NS}}) F \Psi + F \Xi D_{\eta}^{\text{NS}} \llbracket A_d^{\text{NS}}, F \Psi \rrbracket \\ &= D_{\eta}^{\text{NS}} F \Xi (d \Psi - \llbracket A_d^{\text{NS}}, F \Psi \rrbracket - (-)^d \eta \llbracket d, \Xi \rrbracket F \Psi) + \llbracket A_d^{\text{NS}}, F \Psi \rrbracket. \end{aligned}$$

We thus obtain

$$(-)^d d A_{\eta}^{\text{R}} = D_{\eta}^{\text{NS}} A_d^{\text{R}} - \llbracket A_{\eta}^{\text{R}}, A_d^{\text{NS}} \rrbracket$$

where  $A_{\eta}^{\text{R}} \equiv F \Psi$  is an R pure-gauge-like field and an R associated field  $A_d^{\text{R}}$  is defined by

$$A_d^{\text{R}} \equiv F \Xi \left( (-)^d d \Psi + \llbracket A_{\eta}^{\text{R}}, A_d^{\text{NS}} \rrbracket + \eta \llbracket d, \Xi \rrbracket A_{\eta}^{\text{R}} \right).$$

### The original form of Kunitomo-Okawa's action

The original form of Kunitomo-Okawa's complete action is

$$S = -\frac{1}{2} \langle \xi Y \Psi, Q \Psi \rangle - \int_0^1 dt \langle A_t^{\text{NS}}, Q A_{\eta}^{\text{NS}} + m_2(A_{\eta}^{\text{R}}, A_{\eta}^{\text{R}}) \rangle,$$

where  $A_{\eta}^{\text{R}} = F(t) \Psi^{\text{R}}$ . Note that  $F(t)$  satisfies  $F(t=0) = 0$  and  $F(t=1) = F$ . Using the cyclic property of the star product  $m_2$ ,  $A^{\text{R}} = D_{\eta}^{\text{NS}} F \Xi A_{\eta}^{\text{R}}$ , and  $\llbracket D_{\eta}^{\text{NS}}, F \Xi \rrbracket = 1$ , we find

$$\begin{aligned} -\int_0^1 dt \langle A_t^{\text{NS}}, m_2(A_{\eta}^{\text{R}}, A_{\eta}^{\text{R}}) \rangle &= \frac{1}{2} \int_0^1 dt \langle A_{\eta}^{\text{R}}, \llbracket A_t^{\text{NS}}, A_{\eta}^{\text{R}} \rrbracket \rangle = \frac{1}{2} \int_0^1 dt \langle D_{\eta}^{\text{NS}} F \Xi \Psi, \llbracket A_t^{\text{NS}}, A_{\eta}^{\text{R}} \rrbracket \rangle \\ &= \frac{1}{2} \int_0^1 dt \langle \Psi, F \Xi D_{\eta}^{\text{NS}} \llbracket A_t^{\text{NS}}, A_{\eta}^{\text{R}} \rrbracket \rangle = \frac{1}{2} \int_0^1 dt \langle \Psi, \partial_t (F \Psi) \rangle \\ &= \frac{1}{2} \langle \Psi, F \Psi \rangle = -\frac{1}{2} \langle \xi Y \Psi, \eta X F \Psi \rangle. \end{aligned}$$

Note also that  $XY \Psi^{\text{R}} = \Psi$  and  $\eta \xi \Psi = \Psi^{\text{R}}$ . Hence, we find that

$$-\frac{1}{2} \langle \xi Y \Psi, Q \Psi \rangle - \int_0^1 dt \langle A_t^{\text{NS}}, m_2(F \Psi, F \Psi) \rangle = -\frac{1}{2} \langle \xi Y \Psi, Q \Psi + \eta X F \Psi \rangle = -\frac{1}{2} \langle \xi Y \Psi, Q F \Psi \rangle.$$

As we explained in section 1.1, this is equal to (1.1).



### Identification of $A_t \cong \tilde{A}_t$ provides $A_\eta = \tilde{A}_\eta$

We check that the identification  $A_t = A_t^{\text{NS}} + A_t^{\text{R}}$  and  $\tilde{A}_t = \tilde{A}_t^{\text{NS}} + \tilde{A}_t^{\text{R}}$  provide a field redefinition of  $(\Phi^{\text{NS}}, \Psi^{\text{R}})$  and  $(\tilde{\Phi}^{\text{NS}}, \tilde{\Psi}^{\text{R}})$  with  $\tilde{\Psi}^{\text{NS}} = \eta \tilde{\Phi}^{\text{NS}}$ . We start is  $A_t - \tilde{A}_t = 0$ . Then the relation  $\eta(A_t - \tilde{A}_t) = 0$  automatically holds. Recall that we have WZW-like relation  $\partial_t A_\eta = D_\eta A_t$  and  $\partial_t \tilde{A}_\eta = \tilde{D}_\eta A_t$  where  $D_\eta B = \eta B - m_2|_0(A_\eta, B) - (-)^{A_\eta B} m_2|_0(B, A_\eta)$ . Therefore, using these WZW-like relations and the identification  $A_t = \tilde{A}_t$ , one can rewrite  $\eta(A_t - \tilde{A}_t) = 0$  as

$$\partial_t(A_\eta - \tilde{A}_\eta) = m_2|_0(A_\eta - \tilde{A}_\eta, A_t) - m_2|_0(A_t, A_\eta - \tilde{A}_\eta). \quad (\text{A.1})$$

For brevity, we define  $\mathcal{I}^{\text{NS}}(t) \equiv A_\eta^{\text{NS}}(t) - \tilde{A}_\eta^{\text{NS}}(t)$  and  $\mathcal{I}^{\text{R}}(t) \equiv A_\eta^{\text{R}}(t) - \tilde{A}_\eta^{\text{R}}(t)$ . Note that the  $t = 0$  values  $A_\eta(t = 0) = \tilde{A}_\eta(t = 0) = 0$  gives the initial conditions  $\mathcal{I}^{\text{NS}}(t = 0) = 0$  and  $\mathcal{I}^{\text{R}}(t = 0) = 0$ . Then, the NS output state and R output state of (A.1) can be separated as

$$\text{NS : } \quad \frac{\partial}{\partial t} \mathcal{I}^{\text{NS}}(t) = \llbracket \mathcal{I}^{\text{NS}}(t), A_t^{\text{NS}}(t) \rrbracket, \quad (\text{A.2})$$

$$\text{R : } \quad \frac{\partial}{\partial t} \mathcal{I}^{\text{R}}(t) = \llbracket \mathcal{I}^{\text{R}}(t), A_t^{\text{NS}}(t) \rrbracket + \llbracket \mathcal{I}^{\text{NS}}(t), A_t^{\text{R}}(t) \rrbracket. \quad (\text{A.3})$$

The initial condition  $\mathcal{I}^{\text{NS}}(t = 0) = 0$  provides the solution  $\mathcal{I}^{\text{NS}}(t) = 0$  of the differential equation (A.2), which means  $A_\eta^{\text{NS}} = \tilde{A}_\eta^{\text{NS}}$ . Then, the R output equation (A.3) reduces to

$$\text{R : } \quad \frac{\partial}{\partial t} \mathcal{I}^{\text{R}}(t) = \llbracket \mathcal{I}^{\text{R}}(t), A_t^{\text{NS}}(t) \rrbracket$$

and the initial condition  $\mathcal{I}^{\text{R}}(t = 0) = 0$  also provides the solution  $\mathcal{I}^{\text{R}}(t) = 0$ , which implies  $A_\eta^{\text{R}} = \tilde{A}_\eta^{\text{R}}$ . As a result, under the identification  $A_t \cong \tilde{A}_t$ , we obtain  $A_\eta = \tilde{A}_\eta$ .

### Field relation of $(\Phi^{\text{NS}}, \Psi^{\text{R}})$ and $(\tilde{\Phi}^{\text{NS}}, \tilde{\Psi}^{\text{R}})$

Under the identification  $A_t \cong \tilde{A}_t$ , we obtained  $A_\eta = \tilde{A}_\eta$ , which provides

$$\frac{1}{1 - A_\eta} \otimes A_t \otimes \frac{1}{1 - A_\eta} = \hat{\mathbf{G}} \frac{1}{1 - \tilde{\Psi}} \otimes \xi_t \tilde{\Psi} \otimes \frac{1}{1 - \tilde{\Psi}}, \quad (\text{A.4})$$

where  $\tilde{\Psi} = \eta \tilde{\Phi}^{\text{NS}} + \tilde{\Psi}^{\text{R}}$  and  $\xi_t = \xi \partial_t$ . One can read the NS and R outputs of (A.4) as

$$\text{NS : } \quad \tilde{\Phi}^{\text{NS}} = \pi_1 \int_0^1 dt \left( \hat{\mathbf{G}}^{-1} \frac{1}{1 - A_\eta(t)} \otimes A_t(t) \otimes \frac{1}{1 - A_\eta(t)} \right) \Big|_{\text{NS}}, \quad (\text{A.5})$$

$$\text{R : } \quad \tilde{\Psi}^{\text{R}} = \pi_1 \eta \int_0^1 dt \left( \hat{\mathbf{G}}^{-1} \frac{1}{1 - A_\eta(t)} \otimes A_t(t) \otimes \frac{1}{1 - A_\eta(t)} \right) \Big|_{\text{R}}. \quad (\text{A.6})$$

## References

- [1] H. Kunitomo and Y. Okawa, “Complete action of open superstring field theory,” arXiv:1508.00366 [hep-th].
- [2] T. Erler, S. Konopka and I. Sachs, “Ramond Equations of Motion in Superstring Field Theory,” arXiv:1506.05774 [hep-th].
- [3] T. Erler, Y. Okawa and T. Takezaki, “ $A_\infty$  structure from the Berkovits formulation of open superstring field theory,” arXiv:1505.01659 [hep-th].

- [4] T. Erler, “Relating Berkovits and  $A_\infty$  Superstring Field Theories; Small Hilbert Space Perspective,” arXiv:1505.02069 [hep-th].
- [5] T. Erler, “Relating Berkovits and  $A_\infty$  Superstring Field Theories; Large Hilbert Space Perspective,” arXiv:1510.00364 [hep-th].
- [6] K. Goto and H. Matsunaga, “On-shell equivalence of two formulations for superstring field theory,” arXiv:1506.06657 [hep-th].
- [7] S. Konopka, “The S-Matrix of superstring field theory,” arXiv:1507.08250 [hep-th].
- [8] A. Sen and E. Witten, “Filling the gaps with PCOs,” JHEP **1509** (2015) 004 [arXiv:1504.00609 [hep-th]].
- [9] A. Sen, “Supersymmetry Restoration in Superstring Perturbation Theory,” arXiv:1508.02481 [hep-th].
- [10] A. Sen, “BV Master Action for Heterotic and Type II String Field Theories,” arXiv:1508.05387 [hep-th].
- [11] K. Goto and H. Matsunaga, To appear.
- [12] N. Berkovits, “SuperPoincare invariant superstring field theory,” Nucl. Phys. B **450** (1995) 90 [Erratum-ibid. B **459** (1996) 439] [hep-th/9503099].
- [13] N. Berkovits, “A New approach to superstring field theory,” Fortsch. Phys. **48** (2000) 31 [hep-th/9912121].
- [14] Y. Okawa and B. Zwiebach, “Heterotic string field theory,” JHEP **0407** (2004) 042 [hep-th/0406212].
- [15] N. Berkovits, Y. Okawa and B. Zwiebach, “WZW-like action for heterotic string field theory,” JHEP **0411** (2004) 038 [hep-th/0409018],
- [16] H. Matsunaga, “Construction of a Gauge-Invariant Action for Type II Superstring Field Theory,” arXiv:1305.3893 [hep-th].
- [17] H. Matsunaga, “Nonlinear gauge invariance and WZW-like action for NS-NS superstring field theory,” JHEP **1509** (2015) 011 [arXiv:1407.8485 [hep-th]].
- [18] N. Berkovits, “The Ramond sector of open superstring field theory,” JHEP **0111** (2001) 047 [hep-th/0109100].
- [19] Y. Michishita, “A Covariant action with a constraint and Feynman rules for fermions in open superstring field theory,” JHEP **0501** (2005) 012 [hep-th/0412215].
- [20] H. Kunitomo, “The Ramond Sector of Heterotic String Field Theory,” PTEP **2014** (2014) 4, 043B01 [arXiv:1312.7197 [hep-th]],

- [21] H. Kunitomo, “First-Order Equations of Motion for Heterotic String Field Theory,” arXiv:1407.0801 [hep-th].
- [22] H. Kunitomo, “Symmetries and Feynman rules for the Ramond sector in open superstring field theory,” PTEP **2015**, no. 3, 033B11 (2015) [arXiv:1412.5281 [hep-th]].
- [23] T. Erler, S. Konopka and I. Sachs, “Resolving Witten’s superstring field theory,” JHEP **1404** (2014) 150 [arXiv:1312.2948 [hep-th]].
- [24] T. Erler, S. Konopka and I. Sachs, “NS-NS Sector of Closed Superstring Field Theory,” arXiv:1403.0940 [hep-th].
- [25] E. Witten, “Interacting Field Theory of Open Superstrings,” Nucl. Phys. B **276** (1986) 291.
- [26] C. Wendt, “Scattering Amplitudes and Contact Interactions in Witten’s Superstring Field Theory,” Nucl. Phys. B **314** (1989) 209.
- [27] Y. Kazama, A. Neveu, H. Nicolai and P. C. West, “Symmetry Structures of Superstring Field Theories,” Nucl. Phys. B **276** (1986) 366.
- [28] H. Terao and S. Uehara, “Gauge Invariant Actions and Gauge Fixed Actions of Free Superstring Field Theory,” Phys. Lett. B **173** (1986) 134.
- [29] J. P. Yamron, “A Gauge Invariant Action for the Free Ramond String,” Phys. Lett. B **174** (1986) 69.
- [30] T. Kugo and H. Terao, “New Gauge Symmetries in Witten’s Ramond String Field Theory,” Phys. Lett. B **208** (1988) 416.
- [31] E. Witten, “Noncommutative Geometry and String Field Theory,” Nucl. Phys. B **268**, 253 (1986).
- [32] E. Getzler and J. D. S. Jones,  $A_\infty$ -algebras and the cyclic bar complex, Illinois J. Math. **34**, 256 (1990).
- [33] M. Penkava and A. S. Schwarz,  $A_\infty$  algebras and the cohomology of moduli spaces, Trans. Amer. Math. Soc. **169**, 91 (1995) [hep-th/9408064].
- [34] H. Kajiura, “Noncommutative homotopy algebras associated with open strings,” Rev. Math. Phys. **19** (2007) 1 [math/0306332 [math-qa]].
- [35] E. P. Verlinde and H. L. Verlinde, “Multiloop Calculations in Covariant Superstring Theory,” Phys. Lett. B **192** (1987) 95.
- [36] E. D’Hoker and D. H. Phong, “The Geometry of String Perturbation Theory,” Rev. Mod. Phys. **60** (1988) 917.
- [37] R. Saroja and A. Sen, “Picture changing operators in closed fermionic string field theory,” Phys. Lett. B **286** (1992) 256 [hep-th/9202087].

- [38] A. Belopolsky, “Picture changing operators in supergeometry and superstring theory,” hep-th/9706033.
- [39] E. Witten, “Superstring Perturbation Theory Revisited,” arXiv:1209.5461 [hep-th].
- [40] B. Jurco and K. Muenster, “Type II Superstring Field Theory: Geometric Approach and Operadic Description,” JHEP **1304** (2013) 126 [arXiv:1303.2323 [hep-th]].
- [41] A. Sen, “Gauge Invariant 1PI Effective Action for Superstring Field Theory,” JHEP **1506** (2015) 022 [arXiv:1411.7478 [hep-th]].
- [42] A. Sen, “Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector,” JHEP **1508** (2015) 025 [arXiv:1501.00988 [hep-th]].
- [43] I. A. Batalin and G. A. Vilkovisky, “Gauge Algebra and Quantization,” Phys. Lett. B **102** (1981) 27.
- [44] I. A. Batalin and G. A. Vilkovisky, “Quantization of Gauge Theories with Linearly Dependent Generators,” Phys. Rev. D **28** (1983) 2567 [Erratum-ibid. D **30** (1984) 508].
- [45] A. S. Schwarz, “Geometry of Batalin-Vilkovisky quantization,” Commun. Math. Phys. **155** (1993) 249 [hep-th/9205088].
- [46] N. Berkovits, “Constrained BV Description of String Field Theory,” JHEP **1203**, 012 (2012) [arXiv:1201.1769 [hep-th]].
- [47] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” Nucl. Phys. B **390**, 33 (1993) [arXiv:hep-th/9206084].
- [48] M. R. Gaberdiel and B. Zwiebach, “Tensor constructions of open string theories. 1: Foundations,” Nucl. Phys. B **505**, 569 (1997) [arXiv:hep-th/9705038].
- [49] Work in progress.
- [50] T. Erler, Y. Okawa, and T. Takezaki, To appear.